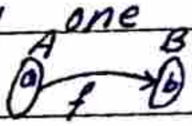
Day: Lecture # 23. Date.

Chapter # 03:- (Continuity)

* Limit
* Function

Definition of Function:-

A functio 'f' from A to B is a relation between A and B such that for each $a \in A$ there is one and only one associated $b \in B$.



The set A is called the Domain of the function and B is Range.

$A = \{a, b\}$ $B = \{c, d\}$

$f(A \text{ and } B) = \{(\underline{a}, c), (\underline{b}, d)\}$

Domain = $\{a, b\}$, Range = $\{c, d\}$

$\longleftrightarrow$

function is denoted as

$$y = f(x)$$

A Real valued function of a Real variable is a function whose Domain and Range are both subset of real Numbers.

Day: Date:

* Limits:-

A value that a function approaches the output for the given input values.

Limits are essential to calculus and mathematical analysis, and used to define continuity, derivatives and Integrals.

Example:-

we can say $f(x)$ approaches at the limit L as 'x' approaches to 'x_0',

we can write as

$\lim_{x \to x_0} f(x) = L$ for $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

$$|x - x_0| < \delta$$

$$|x - x_0| > 0.$$

Example:-

for the function $f(x) = cx$. $c \in R$. Show that

$$\lim_{x \to x_0} f(x) = c x_0$$

Day: Date:

Sol Since , $f(x) = cx_0$

we, can write

$$|f(x) - cx_0| = |cx - cx_0|$$
$$= |c||x - x_0|$$

$c \neq 0$, then

$$|f(x) - cx_0| < \varepsilon$$

and $|x - x_0| < \delta$

where δ is any number such that

$$0 < \delta < \varepsilon .$$

Example:-

Prove that $\lim_{x \to 2} (3x-5) = 1$

Sol

$$L.H.S = \lim_{x \to 2} (3x-5)$$
$$= 3(2) - 5$$
$$= 6 - 5 = 1 = R.H.S$$ Hence Prove!

Example for the the function

$$f(x) = x \sin \frac{1}{x} \qquad x \neq 0$$

Show that $\lim_{x \to 0} f(x) = 0$

Sol

for $f(0) = 0 \sin \frac{1}{0}$

$$= 0 \sin \infty$$

it is undefined

at 0.

Day: Date:

Such as $x_0 = 0$ is undefined
but if
$$0 < |x| < \delta = \varepsilon$$
Prepared by
Fatima Ali

Then
$$|f(x) - 0| = |x \sin \tfrac{1}{x}|$$
$$\leq |x| < \varepsilon$$
$$\delta = \varepsilon \quad \text{or} \quad 0 < \delta < \varepsilon$$
These are possibilities.
$\sin \frac{1}{x}$ has no limit at
$x = 0$.

Uniqueness of Limit :-

Theorem:- If $\lim\limits_{x \to x_0} f(x)$ exists,
Then it is unique.

Proof By Contradiction
let
$$\lim_{x \to x_0} f(x) = L_1 \ \& \ \lim_{x \to x_0} f(x) = L_2$$
Show that $L_1 = L_2$
for every $\varepsilon/2 > 0$
There exists δ_1
$$|f(x) - L_1| < \varepsilon/2$$
$$|x - x_0| < \delta_1$$
and $\varepsilon/2 \ \exists \ \delta_2$
$$|f(x) - L_2| < \varepsilon/2$$
$$|x - x_0| < \delta_2$$

Day: ________ Date: ________

$\delta = \min(\delta_1, \delta_2)$ will satisfy the definition

$$|L_1 - L_2| = |L_1 - L_2 + f(x) - f(x)|$$
$$\leq |f(x) - L_1| + |f(x) - L_2|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$\leq \varepsilon$$

$|L_1 - L_2| < \varepsilon$ where $\varepsilon > 0$

Since

$$L_1 = L_2$$

if the limit of the real valued function exists then it is unique.

Example:- Find limit of

$$\lim_{x\to 2} \frac{9-x^2}{x+1}$$

Sol

$$\lim_{x\to 2} \frac{9-x^2}{x+1}$$

$$= \frac{9-2^2}{2+1} = \frac{9-4}{3} = \frac{5}{3}$$

and $\lim_{x\to 2} (9-x^2)(x+1)$

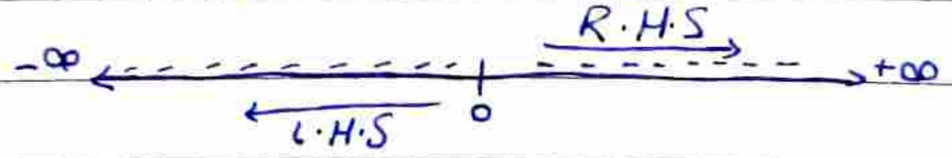
$$= \lim_{x\to 2}(9-x^2) \lim_{x\to 2}(x+1)$$

$$= (9-2^2)(2+1) = (9-4)\cdot 3$$

$$= 5\cdot 3 = 15$$

One Sided Limits:-

Real number line



if $f(x)$ approaches to the left hand limit L as x approaches x_0 from left

$$\lim_{x\to x_0^-} f(x) = L$$

if 'f' is defined on some open interval (a, x_0) and, for each $\varepsilon > 0$

then there exists $\delta > 0$

$$|f(x) - L| < \varepsilon \text{ if}$$

$$|x_0 - \delta| < x < x_0$$

R.H. Limit

if $f(x)$ approaches the Right hand limit L as x approaches x_0 from Right,

$$\lim_{x \to x_0^+} f(x) = L$$

if 'f' is defined on some open intervals (x_0, b)

$$\varepsilon > 0 \quad \exists \quad \delta > 0$$

$$|f(x) - L| < \varepsilon$$

if $\quad x_0 < x < x_0 + \delta$

Example:-

Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$

Sol

As the given function is R.H.

$f(x) = \sqrt{x}$, $L = 0$

by the definition

$$|\sqrt{x} - 0| \leq |\sqrt{x}| \leq |x|$$

$$< \varepsilon = \delta$$

So the interval is $(0, \delta)$

$$\delta = \varepsilon$$

$$0 < x < \delta = \varepsilon$$

Day: ________

Prepared by
Fatima Ali

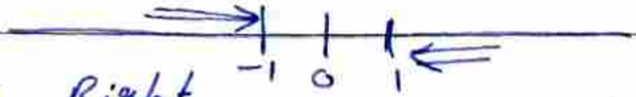
Example:- Show that

$$\lim_{x\to 0^+}\left(\frac{|x|}{x}+x\right) = 1$$

&

$$\lim_{x\to 0^-}\left(\frac{|x|}{x}+x\right) = -1$$

Sol Since for Right and left



For Right

$$f(x) = \frac{|x|}{x} + x \quad \text{for } x > 0$$

$$= \frac{x}{x} + x = 1 + x$$

$f(x) = 1 + x$, $L = 1$

$|1+x-1| \leq |x| < \varepsilon$, $\delta = \varepsilon$

$$0 < x < \delta = \varepsilon$$

For left Hand

$$f(x) = \frac{-x}{x} + x$$

$$f(x) = -1 + x$$

$$L = -1$$

By def.

$$|-1+x-(-1)| = |-1+x+1|$$

$$= |x| < \varepsilon$$

$$-\delta < x < 0$$

Limit at $\pm\infty$ (Infinity)

If f is defined on an interval (a, ∞), then $f(x)$ approaches the limit L as x approaches ∞

$$\lim_{x\to\infty} f(x) = L$$

for every $\varepsilon > 0 \quad \exists\ \beta$

$$|f(x) - L| < \varepsilon \quad \text{if } x > \beta$$

Example let

$$f(x) = 1 - \frac{1}{x^2}$$

$$g(x) = \frac{2|x|}{1+x}$$

and $h(x) = \sin x$

Find limit when $x \to \pm\infty$.

Sol let

$$\lim_{x\to\infty} f(x) = 1$$

So, $L = 1$ & $f(x) = 1 - \frac{1}{x^2}$

By definition

$$|f(x) - 1| = \left|1 - \frac{1}{x^2} - 1\right|$$

$$= \left|-\frac{1}{x}\right| = \frac{1}{x^2} < \varepsilon$$

sqr. root both side

$$\sqrt{\frac{1}{x^2}} < \sqrt{\varepsilon} \Rightarrow \frac{1}{x} > \sqrt{\varepsilon} \Rightarrow x > 1/\sqrt{\varepsilon}$$

ay: ______ Date: ______

$$x > \frac{1}{\sqrt{\varepsilon}} = \beta$$

Prepared by Fatima Ali

(ii) $\lim_{x\to\infty} g(x) = 2$

$$g(x) = \frac{2|x|}{1+x}, \quad L = 2$$

By def.

$$|g(x) - L| = \left| \frac{2|x|}{1+x} - 2 \right|$$

$$= \left| \frac{\cancel{2x} - 2 - \cancel{2x}}{1+x} \right| = \left| \frac{-2}{1+x} \right|$$

$$< \frac{2}{1+x} < \frac{2}{x} < \varepsilon$$

$$x > 2/\varepsilon = \beta$$

(iii) $h(x) = \sin x$

$\lim_{n\to\infty} h(x)$ it does not exists for $\sin x$ if x approaches to ∞.

Day: Date:

Infinite Limits:-

When $f(x)$ approaches to ∞ as x approaches to x_0 from the left, Then

$$\lim_{x \to x_0^-} f(x) = \infty$$

(a, x_0), for every real number M $\exists$ $\delta > 0$,

$$f(x) > M \text{ if}$$

$$x_0 - \delta < x < x_0$$

Similarly

Prepared by
Fatima Ali

$$\lim_{x \to x_0^-} f(x) = -\infty$$

$$\lim_{x \to x_0^+} f(x) = \pm\infty$$

also we can define

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x^2} = \lim_{x \to 0^+} \frac{1}{x^2} = \infty$$

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$\lim_{x \to \infty} x^2 = \lim_{x \to -\infty} x^2 = \infty$$

Day: Date:

Example $f(x) = e^{2x} - e^{x}$

Sol we cannot obtained $\lim\limits_{x \to \infty} f(x)$
s. that

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{2x} - \lim_{x \to \infty} e^{x}$$

$$= \infty - \infty$$

which is undefined or intermediate form.
By simplifying

$$f(x) = e^{2x}(1 - e^{-x})$$
$$= e^{2x}\left(1 - \frac{1}{e^{x}}\right)$$

$$\lim_{x \to \infty} f(x) = \left(\lim_{x \to \infty} e^{2x}\right) \lim_{x \to \infty} \left(1 - \frac{1}{e^{x}}\right)$$

$$= \infty \left(1 - \frac{1}{\infty}\right)$$

$$= \infty(1 - 0) = \infty(1)$$

$$= \infty$$

Day: Lecture #24 (mod- 114 Date 17)
Chapter #3

- Continuity (Informal)

Definition:-

The function 'f' is continuous at x_0 if 'f' is defined on an open interval (a,b) containing x_0,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

* The function f is continuous from left at x_0, defined on an open interval (a, x_0)

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

Prepared by
Fatima Ali

* at Right hand side f is defined on an open interval (x_0, b)

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

- Continuity (Formal)

A function f is continuous at x_0 if and only if it is defined on an open interval (a,b) containing x_0, and for every $\varepsilon > 0$ there is $\delta > 0$

Day: Date:

$|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$

* The function f is continuous from right at x_0, at on an ~~open~~ interval. $[x_0, b)$ for every $\varepsilon > 0$ $\exists$ $\delta > 0$

$$|f(x) - f(x_0)| < \varepsilon$$

$$x_0 \leq x < x_0 + \delta$$

* The function f is continuous from left at x_0, at an interval $(a, x_0]$ for every $\varepsilon > 0$ $\exists$ $\delta > 0$

$$|f(x) - f(x_0)| < \varepsilon$$

$$x_0 - \delta < x \leq x_0$$

Example:

Discuss the continuity of 'f' defined on $[0, 2]$ by

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ x+1 & 1 \leq x \leq 2 \end{cases}$$

Sol

If $x > 0$, $x_0 < 1$, then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |(x - x_0)(x + x_0)|$$

$$= |x + x_0||x - x_0|$$

$$\leq 2|x - x_0| < \varepsilon$$

$$|x - x_0| < \varepsilon/2$$

Hence f is continuous at $(0, 1)$

Day: Date:

If $x > 1$, $x < 2$, then

$$|f(x) - f(x_0)| = |(x+1) - (x_0+1)|$$
$$= |x + 1 - x_0 - 1|$$
$$= |x - x_0| < \varepsilon$$

Hence f is continuous at $(1, 2)$

Example:- Discuss the continuity of $f(x) = \sqrt{x}$, $0 \leq x < \infty$

Sol

Consider

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon$$

Prepared by Fatima Ali

Taking sqr. $\Rightarrow$ $0 \leq x < \varepsilon^2$

So, $\lim\limits_{x \to 0^+} f(x) = f(0)$

if $x_0 > 0$ and $x \geq 0$

Then

$$|f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}|$$

By rationalization

$$= \left| \sqrt{x} - \sqrt{x_0} \times \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \right|$$

$$= \left| \frac{(\sqrt{x})^2 - (\sqrt{x_0})^2}{\sqrt{x} + \sqrt{x_0}} \right|$$

$$= \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| \leq \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon$$

$< \varepsilon$ if $|x - x_0| < \varepsilon \sqrt{x_0}$

Day: Date:

So, $\lim_{x \to x_0} f(x) = f(x_0)$

Hence f is continuous on $[0, \infty)$

- **Piecewise Continuous Function:-**

A function f is Piecewise continuous function on $[a,b]$ if

- $\lim_{x \to x_0^+} f(x)$ exists for all $[a,b)$
- $\lim_{x \to x_0^-} f(x)$ exists for all $(a,b]$
- $\lim_{x \to x_0^\pm} f(x) = f(x_0)$ for all but finitely many points x_0, in (a,b).

In Simple...

(Piecewise are those function which are defined in subset of real number)

Example:- Discuss the following function

$$\begin{cases} 1, & x = 0 \\ x, & 0 < x < 1 \\ 2, & x = 1 \\ x, & 1 < x \leq 2 \\ -1, & 2 < x < 3 \\ 0, & x = 3 \end{cases}$$

Sol at $x = 0$ $f(0) = 1$

at $0 < x < 1$ $f(x) = x$

$x = 1$, $f(1) = 2$

$1 < x \leq 2$ $f(x) = x$

Day: Date:

at $2 < x < 3$ $f(x) = -1$

at $x = 3$ $f(3) = 0$

So, the function is defined on 0 to 3 such as $[0, 3]$

$a = 0$, $b = 3$

- $\lim_{x \to 0^+} f(x)$ at right,

- $\lim_{x \to 3^-} f(x)$ at left

- $\lim_{x \to x_0} f(x) = f(x_0)$ finitely many points.

So, the limit at Right side

$\lim_{x \to 0^+} f(x) = 0 \neq 1$

So, This is not a Right continuous function.

$\lim_{x \to 1^-} f(x) = 1 \neq 2$

$\lim_{x \to 1^+} f(x) = 1$

and for last point

$\text{Limit}_{x \to 3} f(x) = 1$

So, this is piecewise continuous function because it discontinuous at the finitely many points.

Lecture no 25.

Day: Date:

Example:- Discuss the continuity of the following

$$r(x) = \frac{9 - x^2}{x+1}$$

Sol

let $r(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

$f(x) = 9 - x^2$

$g(x) = x + 1$

But at $x + 1 = 0 \Rightarrow x = -1$

$g(x) = 0$ at $x = -1$

So the function is continuous at all else $x = -1$.

Theorem:-

Statement:- If f and g are continuous an set S, then so, are, $f+g$, $f-g$, fg and f/g are continuous at x_0 in S. such that $g(x_0) \neq 0$.

Proof

let S be closed Interval,

$S = [a, b]$, $b = (\text{Sup})\ M.$, $M > 0$

$f(x) < M$ and $g(x) < M$

Day: Date:

Since the function f and g are continuous at x_0, such that

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\lim_{x \to x_0} g(x) = g(x_0)$$

By the definition for every $\varepsilon > 0$ $\exists \delta_1 > 0$

$$|f(x) - f(x_0)| < \varepsilon/2$$

$$|x - x_0| < \delta_1$$

and

$$|g(x) - g(x_0)| < \varepsilon/2$$

$$|x - x_0| < \delta_2$$

i) $f+g$

$$\lim_{x \to x_0} (f+g)(x) = (f+g)x_0$$

By the definition

$$|(f+g)x - (f+g)x_0| = |f(x) + g(x) - f(x_0) - g(x_0)|$$

$$= |f(x) - f(x_0) + g(x) - g(x_0)|$$

$$\leq |f(x) - f(x_0)| + |g(x) - g(x_0)|$$

$$< \varepsilon/2 + \varepsilon/2 < 2\varepsilon/2$$

$$< \varepsilon$$

Hence $f+g$ is continuous at x_0.

ii) $(f-g)$

$$\lim_{x \to x_0} (f+g)x = (f-g)x_0$$

By def. for each $\varepsilon > 0$ $\exists \delta_1 > 0$

Day: Date:

$|(f-g)x-(f-g)x_0| = |f(x)-g(x)-(f(x_0)-g(x_0))|$

$= |f(x)-g(x)-f(x_0)+g(x_0)|$

$= |f(x)-f(x_0)+(-g(x)+g(x_0))|$

$\leq |f(x)-f(x_0)|+|-(g(x)-g(x_0)|$

$< \varepsilon/2 + \varepsilon/2 < 2\varepsilon/2$

$< \varepsilon$

Hence $f-g$ is continuous at x_0.

iii) fg $\qquad |(fg)x-(fg)x_0| < \varepsilon/2M$ By C. Property.

$\lim_{x\to x_0} (fg)x = (fg)x_0$

By the definition

$|(fg)x-(fg)x_0| = |f(x)g(x)-f(x_0)g(x_0)|$

$= |f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)-f(x_0)g(x_0)|$

$= |f(x)(g(x)-g(x_0))+g(x)(f(x)-f(x_0))|$

$\leq |f(x)(g(x)-g(x_0))|+|g(x)(f(x)-f(x_0))|$

$< \cancel{M}\cdot \varepsilon/2\cancel{M} + \cancel{M}\cdot\varepsilon/2\cancel{M}$

$< \varepsilon/2 + \varepsilon/2 < 2\varepsilon/2 < \varepsilon$

$|(fg)x-(fg)(x_0)| < \varepsilon$

Hence Proved!

iv) f/g

By Definition

$\lim_{x\to x_0} (f/g)x = (f/g)x_0$ where $g(x_0)\neq 0$

$\forall\ \varepsilon>0,$

$|(f/g)x-(f/g)x_0| = \left|\frac{f(x)}{g(x}-\frac{f(x_0)}{g(x_0)}\right|$

Day: Date:

$$= \left| \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)} \right|$$

$$= \left|\frac{1}{g(x)g(x_0)}\right| \left| f(x)g(x_0) - f(x_0)g(x) \right|$$

$$= \left|\frac{1}{g(x)g(x_0)}\right| \left| f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x) \right|$$

$$= \left|\frac{1}{g(x)g(x_0)}\right| \left| g(x_0)\left(f(x) - f(x_0)\right) + f(x_0)\left(g(x_0) - g(x)\right) \right|$$

$$\leq \left|\frac{1}{g(x)g(x_0)}\right| \left| |g(x_0)|\,|f(x) - f(x_0)| + |f(x_0)|\,|\cdot g(x) - g(x_0)| \right|$$

$$\leq \frac{1}{M} \cdot M \varepsilon/2 + \frac{1}{M} \cdot M \varepsilon/2$$

$$< \varepsilon/2 + \varepsilon/2 < \varepsilon$$

Hence Proved!

prepared by
Fatima Ali

Lecture no 26

Day: Date:

Removeable Discontinuity:-

Define:-

Such type of discontinuity at a particular point of the Domain x_0, such that we can remove that discontinuity.

In removeable discontinuity function may or may not defined at x_0, but the limit at both side exists.

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x_0 \neq x_0 \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0 \end{cases}$$

is continuous at x_0.

Example:-

Consider the function

$f(x) = x \sin \frac{1}{x}$

* * *

at $x_0 = 0$

$f(0) = 0 \sin \frac{1}{0}$

undefined

Sol

the function is undefined at $x_0 = 0$.

But $\lim_{x \to 0} f(x) = 0$

$$\lim_{x \to 0} \cancel{\sin} \; x \sin \frac{1}{x} = 0$$

$f(0) = 0 \sin \frac{1}{0}$ undefined

at continuity

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \\ L = 0 & x = 0 \end{cases}$$

By redefining the function $g(x)$ we can remove the discontinuity.

Composition of funtions:-

Suppose that f and g are functions with domains D_f and D_g. then composite function $f \circ g$ is defined as

$$(f \circ g)\, x = f(g(x))$$

Example:-

If $f(x) = \log(x)$

$g(x) = \frac{1}{1-x^2}$

Then $f \circ g = ?$

Sol

By the definition

$$(f \circ g)(x) = f(g(x)$$

$$= \log(g(x))$$

$$(f \circ g)\, x = \log \frac{1}{1-x^2}$$

Similarly:-

$$(g \circ f)\, x = g(f(x))$$

$$= \frac{1}{1-(\log x)^2}$$

Day: Date:

Theorem:-

Suppose g is continuous at x_0, $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$ Then $g \circ f$ is Continuous at x_0.

Proof

Since $g(x_0)$ is an interior point and f is continuous at $g(x_0)$, for every $\varepsilon > 0$, $\exists$ $\delta_1 > 0$ such that $f(t)$ is defined,

$$|f(t) - f(g(x_0))| < \varepsilon$$

if $|t - g(x_0)| < \delta_1$

Since g is continuous at x_0

$$|g(x) - g(x_0)| < \delta_1$$

if $|x - x_0| < \delta$

if g is continuous at x_0 then $f \circ g$ will also be continuous at x_0,

$$|f(g(x)) - f(g(x_0))| < \varepsilon$$

if $|x - x_0| < \delta$

So $f \circ g$ is Continuous at x_0

Hence Proved!

Day: Date:

Example:-

$f(x) = \sqrt{x}$ $\quad g(x) = \dfrac{9-x^2}{x+1}$

$f \circ g = ?$

Sol

Since $g(x)$ is undefined at $x=-1$

So, the function f is continuous for $x>0$, and g is continuous for $x \neq -1$. for all.

$g(x)>0$ if we tak $x<-3$ or $-1<x<3$, the composite function

$$(f \circ g)x = f(g(x))$$
$$= \sqrt{\frac{9-x^2}{x+1}}$$

Since the function is continuous from the left at -3 and 3.

Day: Date:

Bounded function:-

If a function f is bounded above and bounded below on a set S, then the function is bounded.

Bounded below function:-

A function f is bounded below on a set S, if there is a real number m such that

$$f(x) \geq m \quad \text{for all } x \in S.$$

the Set

$$V = \{f(x) : x \in S\}$$

it has infimum α,

$$\alpha = \inf_{x \in S} f(x)$$

Bounded above function:-

A function f is bounded above on set S if there is a real number M, s.that $f(x) \leq M$ for all $x \in S$.

Then the set has supremum β

$$\beta = \sup_{x \in S} f(x)$$

For bounded

$$m \leq f(x) \leq M \quad \text{for all } x \in S.$$

Day: Date:

Example:- The function:-

$$g(x) = \begin{cases} \frac{1}{2}, & x=0 \text{ or } x=1 \\ 1-x, & 0<x<1. \end{cases}$$

Sol

Since $0=x=1$ and $0<x<1$

So, we can write $0 \le x \le 1$

which is bounded at $[0, 1]$,

and

$$\sup_{0\le x\le 1} g(x) = 1$$

and $\inf_{0\le x\le 1} g(x) = 0$.

So, the function has no max. or min. at close interval $[0, 1]$.

Since it does not assume the value of 0 and 1.

Example:- the function

$$h(x) = 1-x, \quad 0 \le x \le 1$$

Sol Since the function is bounded at $[0,1]$. But it has values at $x=0$ and $x=1$

So

$$\max_{0\le x\le 1} h(x) = 1$$

and $\min_{0\le x\le 1} h(x) = 0$.

Example:- function

$$f(x) = e^{x(x-1)} \sin \frac{1}{x(x-1)}, \quad 0<x<1$$

Day: Date:

Sol Since $\sin \frac{1}{x(x-1)}$ oscillate at $x=0$, So the function ascillate b/w $\pm e^{x(x-1)}$.

we have

$$\sup_{0<x<1} f(x) = 1 \text{ and } \inf_{0<x<1} f(x) = -1$$

the function f has no max and min at $(0,1)$.

prepared by
Fatima Ali

Theorem:-

If f is continuous on finite closed interval $[a,b]$, then f is bounded on $[a,b]$.

Proof Suppose $t \in [a,b]$.

Since f is continuous at t, there is an open interval I_t

$$|f(x) - f(t)| < \varepsilon$$

whenever $|x - t| < \delta$ So, this is an open interval s. that

$$t-\delta < x < t+\delta$$

let $I_t = (t-\delta, t+\delta)$ and $\varepsilon = 1$

So

$$|f(x) - f(t)| < 1 \quad \text{if } x \in I_t \cap [a,b].$$

Day: Date:

Let H be a set
$$H = \{ I_t : a \leq t \leq b \}$$
This collect is an open covering of $[a,b]$.

By Heine Borel theorem
the collection $t_1, t_2, t_3 \cdots t_n$
and $I_{t_1} \cup I_{t_2} \cup I_{t_3} \cdots \cup I_{t_n} = [a,b]$

Since $[a,b]$ is compact.
according to the continuity with $t = t_i$
$$|f(x) - f(t_i)| < 1 \quad \text{if} \quad x \in I_{t_i} \cap [a,b]$$
Therefore
$$|f(x)| = |f(x) - f(t_i) + f(t_i)|$$
$$\leq |f(x) - f(t_i)| + |f(t_i)|$$
$$\leq 1 + |f(t_i)| \quad \forall \ x \in I_{t_i} \cap [a,b]$$
Let
$$M = 1 + \max_{1 \leq i \leq n} |f(t_i)|$$

$f(x) \leq M$ So $-M \leq f(x) \leq M$ $\forall [a,b]$

Since the set $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$
we, have
$$|f(x)| \leq M \quad \text{if} \quad x \in [a,b]$$
Hence f is bounded.

Lecture no 27

Day: Date: (127 – 136)

The Intermediate Value Theorem:-

Statement:-

Suppose that f is continuous on $[a,b]$, $f(a) \neq f(b)$, and μ is b/w $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a,b).

Proof

Suppose $f(a) < \mu < f(b)$

The set $\{S = x : a \le x \le b \ \& \ f(x) \le \mu\}$ is bounded and None empty.

Let $c = \text{Sup } S$.

Then we prove that $f(c) = \mu$

If $f(c) > \mu$

Then $c > a$

Since f is continuous at c.

Then there is an $\varepsilon > 0$, s.that

$$f(x) > \mu \quad \text{if} \quad c - \varepsilon < x \le c$$

$c - \varepsilon$ is an upper bound of S, which contradict the definition of c as the sup of S.

If $f(c) < \mu$, then $c < b$

$$f(x) < \mu \quad \text{for} \quad c \le x \le c + \varepsilon$$

So, c is not upper bound of S.

$$f(c) = \mu$$ (Proved!)

Day: # Uniform Continuity Date:

- Continuous function:-

A function f is continuos on a subset S of its domain if for every $\varepsilon > 0$ and each x_0 in S, there is a $\delta > 0$, which may depend upon x_0 as well as ε, s.that

$$|f(x) - f(x_0)| < \varepsilon$$

if $|x - x_0| < \delta$ and $x \in D_f$.

prepared by
Fatima Ali

- Uniformly Continuous:-

A function f is uniformly continuous on a Subset S of its domain if, for each $\varepsilon > 0$ $\exists$ $\delta > 0$, s.that

$$|f(x) - f(x')| < \varepsilon$$

$$|x - x'| < \delta \quad \forall \; x, x' \in S.$$

Example:-

Check the uniform continuity $f(x) = 2x$.

Sol

By the def.

$$|f(x) - f(x')| < \varepsilon$$

$$|f(x) - f(x')| = 2|x - x'| < \varepsilon$$

$$\Rightarrow |x - x'| < \varepsilon/2$$

the function is uniformly continuous on $(-\infty, \infty)$

Day: Date:

Example:- Show that the function $g(x) = x^2$ is uniformly continuous on $[-\gamma, \gamma]$ for any finite γ.

Sol By def.

$$|g(x) - g(x')| = |x^2 - x'^2|$$
$$= |(x - x')(x + x')|$$
$$\leq |x + x'||x - x'|$$
$$\leq 2\gamma |x - x'|$$

So, $|g(x) - g(x')| < \varepsilon$

if $|x - x'| < \delta = \varepsilon/2\gamma$

and $-\gamma \leq x, x' \leq \gamma$.

Example:- The function $g(x) = x^2$ is not uniformly continuous on $(-\infty, \infty)$

Sol

if $\delta > 0$ there are real number x and x' s.that

$$|x - x'| = \delta/2$$

and $|g(x) - g(x')| \geq 1$

By the def.

$$|g(x) - g(x')| = |x^2 - x'^2|$$
$$= |x + x'||x - x'|$$

if $|x - x'| = \delta/2$ and $x, x' > 1/\delta$

Then $|x - x'||x + x'| > \frac{\delta}{2}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) = 1.$

Day: Date:

Example:- the function $f(x) = \cos\frac{1}{x}$ is continuous $(0, 1]$.

Sol

Since at $x = 0$, $\cos\frac{1}{x}$ is undefined, function is not uniformly Continuous on $(0, 1]$.

Prepared by
Fatima Ali

~~Theorem:~~

Monotonic function:-

A function f is nondecreasing on an interval I if

$$f(x_1) \le f(x_2), \quad x_1, x_2 \in I, \quad x_1 < x_2 \quad \text{--- (1)}$$

or f is nonincreasing on I if

$$f(x_1) \ge f(x_2), \quad x_1, x_2 \in I, \quad x_1 < x_2 \quad \text{--- (2)}$$

If we replace '$\le$' by '$<$' in eq (1) then function f is increasing on I. Similarly if we replace $\ge$ by $>$ in (2) Then f is decreasing on I.

In these cases the function f is strictly monotonic on I.

Day: ________ Date: ________

Example:-

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2 \end{cases}$$

Sol

So the function is nondecreasing on an Interval $I = [0, 2]$.

Example:-

As $f(x_1)$ $f(x_2)$ $x_1 < x_2$

for $[0, 1)$ $x_1 < x_2$

for $(1, 2]$ $x_1 < x_2$

The function $g(x) = x^2$ is increasing on $[0, \infty)$

$f(x_1) = x_1^2$ if $x_1 < x_2$

$f(x_2) = x_2^2$ $x_1^2 < x_2$

So, the function is increasing at interval $[0, \infty)$.

The function is $h(x) = -x^3$ is dec. on $(-\infty, \infty)$.

Since $h(x) = -x^3$ is decreasing function at extended real number system.

Lecture no 28

Day: Date:

Limits Inferior & Superior:-

Suppose that f is bounded on $[a, x_0)$, where x_0 may be finite or infinite.

for $a \le x < x_0$ $\qquad x \in (a, x_0)$

$$S_f(x : x_0) = \sup_{x \le t < x_0} f(t)$$

$$I_f(x : x_0) = \inf_{x \le t < x_0} f(t)$$

Then the left limit (Superior) is defined as

$$\lim_{x \to x_0-} \text{Sup}\, f(x) = \lim_{x \to x_0-} S_f(x ; x_0)$$

Left limit inferior at x_0

$$\lim_{x \to x_0-} \text{Inf}\, f(x) = \lim_{x \to x_0-} I_f(x : x_0)$$

Prepared by
Fatima Ali

(If $x_0 = \infty$ then the left limit is defined as $x_{0-} = \infty$)

Theorem:-

If f is bounded on $[a, x_0)$, then $\beta = \lim_{x \to x_0-} \text{Sup}\, f(x)$ exists and is the unique real number with the following properties

Day: Date:

1) If $\varepsilon > 0$, there is an ~~rate~~ a_1 in interval $[a, x_0)$ s.that

$$f(x) > \alpha - \varepsilon \quad \text{if} \quad a_1 \leq x < x_0$$

2) If $\varepsilon > 0$, and a_1 is in $[a, x_0)$,

$$f(\bar{x}) < \alpha + \varepsilon \quad \text{for} \quad \bar{x} \in [a_1, x_0)$$

Inverse of the function:-

Def:- An inverse function is a function that undoes the action of another function. Let a function

$$f(x) = y$$

Then the ~~in~~ inverse of the function is

$$x = f^{-1}(y)$$

For example a relation A s.that

$$A = \{(a,b), (c,d), (e,f)\}$$

then inverse of A is

$$A^{-1} = \{(b,a), (d,c), (f,e)\}$$

Example:-

If $f(x) = x^2 \qquad 0 \leq x \leq R$

Sol

Since Domain of the function is $D_f = [0, R]$

The function f is increasing function

$x = 0$ in $f(x) = x^2$

for $f(0) = 0$

for $x = R$

$$f(R) = R^2$$

So the inverse function will have domain $[0, R^2]$

let y be a real number s.that $0 \le y \le R^2$

let

$$f(x_0) = x_0^2 = y \quad , x_0 \in [0, R^2]$$

$$\Rightarrow \quad x_0^2 = y$$

$$x_0 = \pm\sqrt{y}$$

Since $-\sqrt{y} \notin [0, R^2]$

So $x_0 = \sqrt{y}$

So inverse of the given function

$$f(x) = y$$

$$f^{-1}y = g(y)$$

$$g(y) = \sqrt{y} \quad 0 \le y \le R^2$$

Example:-

Find inverse

$$f(x) = 2x + 4 \quad 0 \le x \le 2$$

Sol Domain of f $[0, 2]$

So,

at $x = 0$ in $f(x) = 2x + 4$

$f(0) = 4$

at $x = 2$

$f(2) = 2(2) + 4 = 8$

So, $y \in [4, 8]$

Let

$f(x) = y$

$2x_0 + 4 = y$

$2x_0 = y - 4$

$x_0 = \frac{y-4}{2}$

So,

inverse of the given function

$f(x) = 2x + 4$

is

$f^{-1}(y) = \frac{y-4}{2}$

Chapter no 3 has been completed...
For more help visit YouTube channel
Professor Chill...

Prepared by Fatima Ali..
Remember in your precious prayers