lecture #01 MTH 403

Introduction (Real and imaginary numbers)

Topic #01:-

Imaginary Number:

An introduction and Applications in daily life.

* Natural and Rational Number are Imaginary number:-
  Example:- $1/2$, $3/4$, 100 —
* Irrational Number => "π" —
* Signal processing (Crowds -> wesipeiog)
* (AC) Imaginary ~~number~~ currents
* $\boxed{Z_{new} = Z^2_{old} + \text{Constant}}$ => **Algorithm**

Topic # 02:- "i" = $90°$

Real Number and their Transformation 1

In early ages Numbers and points are Consider as different thing.
Numbers are Studied in Algebra nd point are Studied in Geometric.

But in present era:-

"Points are nothing but the numbers and numbers are nothing but the points" on the real line".

* Addition is a translation on a real line number

* Multiplication:- (Scaling)

Contraction (making small) | dilation (lengthen)

MCQ'S

(-1) is Half rotation (180°)

$(-1)^2 \rightarrow$ Complete rotation (360°)

Topic :- 03

~~Real Number~~

Imaginary Numbers (definition and its Geometrical representation)

The Numbers on 90° rotation on a real line will be above or blew the real line and that numbers only can be imagin

90° rotation will correspond to imaginary number.

imaginary numbers:-

"Imaginary numbers are nothing but there are a quarter rotation of any number which will transform 90° rotation of any number."

$$\sqrt{-1} = \text{'}i\text{'}$$

Topic:- 04

Integral power of Iota

$$i = \sqrt{-1}$$

$$i^2 = (i \cdot i) = -1$$

$$i^3 = -1$$

$$i^4 = 1$$

$$i^5 = i$$

$$i^{2n} = (i^2)^n = (-1)^n$$

$$= \begin{cases} 1 & n \in E \\ -1 & n \in O \end{cases}$$

Mutiplicative inverse = Additive inverse

[redacted]

Caution (regarding Imaginary number)

Caution :- $x, y \in \mathbb{R}^+$

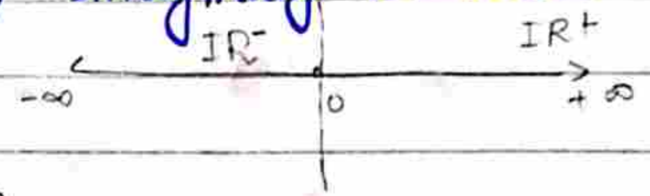
* $\sqrt{x}\sqrt{y} = \sqrt{xy} \Rightarrow$ real Number

But in 'i'

$i = \sqrt{-1}$

$\sqrt{-2} = \sqrt{(-1)(2)} = \sqrt{2}\, i$

Sign of Imaginary Number :-



if $i > 0$

$\Rightarrow i \cdot i > i(0)$

$\Rightarrow i^2 > 0$

$\Rightarrow (-1) > 0$ (a contradiction)

if $i < 0$

$\Rightarrow i \cdot i < i(0)$

$\Rightarrow i^2 < 0$

$\Rightarrow (-1) < 0 \Rightarrow$ a contradiction

* "Square of any number can't be less-than zero

* "Therefore 'i' has no sign*"

Complex number, their algebra, polar and Argand Diagram

Lecture # 02:-

* Complex Numbers:-

A complex number is a Combination of real and Imaginary number.

$$\underset{\text{Real}}{7} + \underset{\text{Imaginary}}{3i}$$

* Argand diagram [Complex number]

A Collection of all the Compl number is called argand ~~dig~~ diagram.

* Algebra of Complex Numbers:-

(i) Equality :- $\mathbb{Z} \cong \mathbb{R}\times\mathbb{R}$

for example we have two Complex number.

$$z_1 = x_1 + iy_1 , \quad z_2 = x_2 + iy_2$$

$$z_1 = z_2 \iff x_1 + iy_1 = x_2 + iy_2$$

$$\left\{ \begin{array}{l} (x_1, y_1) = (x_2, y_2) \\ \Rightarrow x_1 = x_2 , \quad y_1 = y_2 \end{array} \right.$$

$$Re\, z_1 = Re\, z_2 , \quad Im\, z_1 = Im\, z_2$$

• Addition and Subtaraction of Complex number

$$z_1 = x_1 + iy_1 , \quad z_2 = x_2 + iy_2$$

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$= (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) \pm i(y_1 - y_2)$$

* Conjugate of a Complex Number:-

Conjugate is nothing but the reflection of any Complex Number about its reflexsive ~~origion~~ position.

$$z = x + iy$$
$$\bar{z} = x - iy$$

* Multiplication, Division and Related laws:-

- **Mutiplication:-**

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$
$$= x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2 y_1 y_2$$
$$= x_1 x_2 + i(x_1 y_2 + x_2 y_1) + (-1) y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

- **Division:-**

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1 x_2 - i x_1 y_2 + i x_2 y_1 - i^2 y_1 y_2}{(x_2)^2 - (i y_2)^2}$$

$$= \frac{x_1 x_2 - i(x_1 y_2 + x_2 y_1) - (-1) y_1 y_2}{(x_2)^2 - (-1) y_2^2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\left(\frac{x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}\right)$$

*** Related law's:-**

$\cancel{z_1 z_2 = z_2 z_1}$

(i) Commutative law:-

$$z_1 z_2 = z_2 z_1 \quad , \quad z_1 + z_2 = z_2 + z_1$$

(ii) Associative law:-

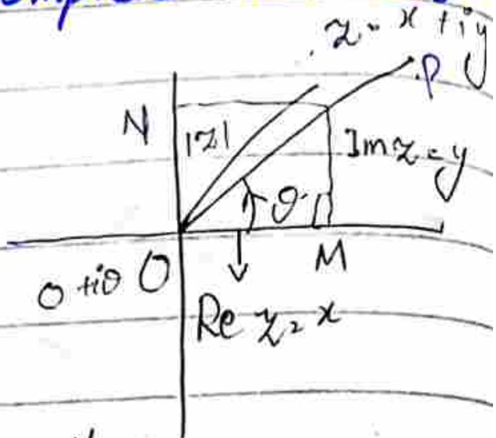
$$z_1 . (z_2 z_3) = (z_1 z_2) . z_3$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

(iii) Distributive law:-

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

• Polar form of a Complex number:-



$$z = x + iy$$

form ⊥ ΔOPM

$$\cos\theta = \frac{OM}{OP} = \frac{Re z}{|z|} = \frac{z}{|z|}$$

$$x = |z| \cos\theta \rightarrow (1)$$

$$\sin\theta = \frac{PM}{OP} = \frac{\cancel{Im}\, z}{|z|} = \frac{y}{|z|}$$

$$y = |z| \sin\theta \quad (2)$$

$$\therefore z = x + iy = |z| \cos\theta + i|z| \sin\theta$$

$$= z(\cos\theta + i\sin\theta)$$

$$z = z \cos\theta \rightarrow (3)$$ polar form of 'z'

from ⊥ ΔOPM, applying Pythagoem theorem

$$|z|^2 = \overline{OP}^2 = \overline{OM}^2 + \overline{PM}^2 = x^2 + y^2 =$$

$$|z|^2 = x^2 + y^2$$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(Re z)^2 + (Im z)^2}$$

from ⊥ OPM

$$\tan\theta = \frac{PM}{OM} = \frac{Im z}{Re z} = \frac{y}{x}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$\boxed{Argu = \theta = \tan^{-1} \frac{y}{x}}$$

Uniqueness of polar form and principle Arguments:-

θ or $\theta \pm \{k(2\pi)$, $k \in \mathbb{Z}$

have same orientation.

$\bar{z} = |z|(\cos\theta + i\sin\theta) = |z|(\cos(\theta + 2k\pi) + i\sin(\theta +$

$k \in \mathbb{Z}$

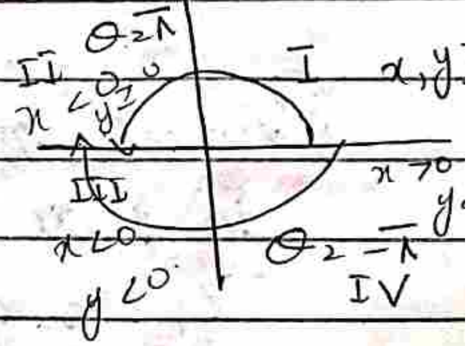
$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right) \pm 2k\pi$

$-\pi < \theta < \pi \pm 2k\pi$

Evalution of principal Arguments:-

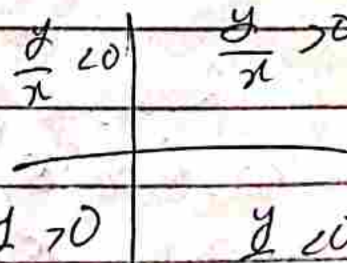
$z = x + iy$

$-\pi < \text{Arg}\ z < \pi$



$\text{Arg}\ z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\tan\theta = \frac{y}{x}$

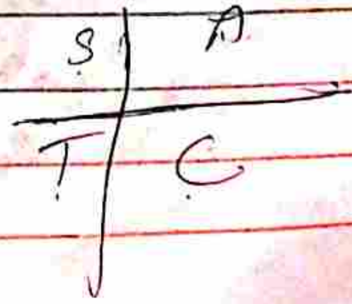


$z = -i + \sqrt{3}i$

$x = -1$, $y = \sqrt{3} > 0$

< 0

* Terminal ray of 'z' lies in IInd quad. *



$Arg\,z = \pi + \tan^{-1}\left(\frac{y}{x}\right) = \pi + \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$
$= \pi + \tan^{-1}(-\sqrt{3})$

$\left\{ \begin{array}{l} \tan^{-1}(-\theta) = -\tan^{-1}\theta \\ \tan^{-1}(\sqrt{3}) = 60^\circ = \pi/3 \end{array} \right.$

$Arg\,z = \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Now; $z = |z|(\cos\theta + i\sin\theta)$ — (i)
$|z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

$z = 2\left(\left(\cos\frac{2\pi}{3}\right) + i\left(\sin\left(\frac{2\pi}{3}\right)\right)\right)$
$= 2\,cis\,(2\pi/3)$

Product of Complex Numbers in polar form:-

$z_1 = r_1\,cis\theta_1\ ,\ z_2 = r_2\,cis\theta_2$
$r_1 = |z_1|,\ r_2 = |z_2|$
$z_1 z_2 = (r_1\,cis\theta_1)(r_2\,cis\theta_2)$
$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= r_1 r_2[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2$
$+ i^2\sin\theta_1\sin\theta_2]$ $\because i^2 = -1$
$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1$
$\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

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$\because \cos\alpha \cdot \cos\beta - \sin\alpha \sin\beta = \cos(\alpha+\beta)$

$\because \sin\alpha \cos\beta + \cos\alpha \sin\beta = \sin(\alpha+\beta)$

$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

By induction :-

$z_1 \cdot z_2 + \cdots + z_n = r_1 r_2 + \cdots r_n [(\cos(\theta_1+\theta_2+\cdots\theta_n)$
$+ i\sin(\theta_1+\theta_2+\cdots\theta_n)]$

Division of Complex Numbers in Polar form:-

$\because z_1 = r_1 \operatorname{cis}\theta_1 \rightarrow z_2 = r_2 \operatorname{cis}\theta_2$

$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis}\theta_1}{r_2 \operatorname{cis}\theta_2}$

$= \frac{r_1(\cos\theta_2 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta}{\cos\theta_2 - i\sin}$

$= \frac{r_1}{r_2} \frac{(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{(\cos\theta_2)^2 - (i\sin\theta_2)^2}$

$= \frac{r_1}{r_2} \frac{[(\cos\theta\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2 c}{\cos^2\theta_2 + \sin^2\theta_2}$

$\therefore \begin{cases} \cos\alpha \cdot \cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) \\ \sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha-\beta) \\ \cos^2\theta + \sin^2\theta_2 = 1 \end{cases}$

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$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

Example:-

Express $\frac{1+\cos 2\theta - i\sin 2\theta}{\cos 2\theta + i\sin 2\theta}$ into $A+iB$.

Solve:-

$$\left\{\begin{array}{l} z_1 = r_1 \operatorname{cis}\theta_2 \\ z_2 = r_2 \operatorname{cis}\theta_2 \\ z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1+\theta_2) \\ \frac{z_1}{z_1} = \frac{r_1}{r_2}(\operatorname{cis}\theta_1 - \theta_2) \\ z = r(\cos\theta + i\sin\theta) \end{array}\right.$$

$$\frac{1+\cos 2\theta - i\sin 2\theta}{\cos 2\theta + i\sin 2\theta}$$

$$\because \left\{\begin{array}{l} 1+\cos 2\theta = 2\cos^2\theta \\ \sin 2\theta = 2\sin\left(\frac{1}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right) \end{array}\right.$$

$$= \frac{2\cos^2\theta - i(2\sin\theta\cdot\cos\theta)}{\cos 2\theta + i\sin 2\theta}$$

$$= \frac{2\cos\theta(\cos\theta - i\sin\theta)}{\cos 2\theta + i\sin 2\theta}$$

$$\left\{\begin{array}{l} -\sin\theta = \sin(-\theta) \\ \cos\theta = \cos(-\theta) \end{array}\right.$$

$$= \frac{2\cos\theta\cdot(\cos(-\theta) + i\sin(-\theta))}{\cos 2\theta + i\sin 2\theta}$$

$$= 2\cos\theta\left[\cos(-\theta+2\theta) + i\sin(-\theta-2\theta)\right]$$

$$= 2\cos\theta\left[\cos(-3\theta) + i\sin(3\theta)\right]$$

$$= 2\cos\theta(\cos 3\theta) - i\sin 3\theta$$

$$= \underbrace{2\cos\theta\cos 3\theta}_{A} + i\underbrace{(-2\cos\theta\sin 3\theta)}_{B}$$

$$= A + iB$$

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Lecture # 03 :-

Conjugates Modulus and locus of a Complex Number.

* Conjugate of a Complex Number and its properties?

(i) $\bar{\bar{z}} = z$

(ii) $z = \bar{z} \iff$ 'z' is pure real

(iii) $z = -\bar{z} \iff$ 'z' is pure imaginary

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Conjugate of Sum = Sum of Conjugate

(v) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(vii) $\overline{(z^n)} = (\bar{z})^n$

(viii) $Re\, z = \frac{z + \bar{z}}{2}$

(ix) $Im\, z = \frac{z - \bar{z}}{2i}$

(x) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2Re(\bar{z}_1 z_2) = 2Re(z_1 \bar{z}_2)$

• Modulus of a Complex Number and its poperties :-

$|Z| = \sqrt{x^2+y^2}$ • order relation

(i) $|z| \geq 0$ (non-negative)

$\because x^2 \geq 0;\ y^2 \geq 0,\ \forall\ x, y \in \mathbb{R}$

$\Rightarrow x^2+y^2 \geq 0 \Rightarrow \sqrt{x^2+y^2} \geq 0$

(ii) $-|z| \leq Re\, z \leq |z|;$

$\Rightarrow -\sqrt{x^2+y^2} \leq x \leq \sqrt{x^2+y^2} \Rightarrow -\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

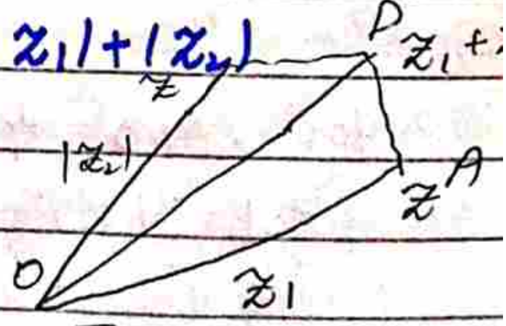
(v) $|z^n| = |z|^n$

(vi) $|z_1 z_2| = |z_1| \cdot |z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

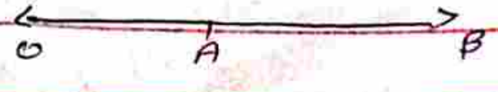
• Triangular Inequality in Complex Numbers :-

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$



$|\overline{OP}| < |\overline{OA}| + |\overline{AP}|$

$\Rightarrow |z_1+z_2| < |z_1| + |z_2|$



$|\overline{OP}| = |\overline{OA}| + |\overline{AP}|$

$|z_1 + z_2| = |z_1| + |z_2|$

Proof:

for R.H.S

$|z|^2 = z \cdot \bar{z}$

$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) \quad \because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$= z_1 \bar{z}_1 + z_1 \bar{z}_2 + \bar{z}_1 z_2 + z_2 \cdot \bar{z}_2$

$\because z\bar{z} = |z|^2$

$= |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1 \bar{z}_2)$

$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$

$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \quad \forall \; x, y \in R, \; x, y \geq$

$|z_1 + z_2| \leq |z_1| + |z_2| \longrightarrow (A)$

It can be extended to

$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_n| + \cdots |z_n|$

$|\sum_{k=1}^{n} z_n| \leq \sum_{j=1}^{n} |z_j|$

for L.H.S & Middle

$|z_1| = |z_1 + z_2 - z_2| = |z_1 + z_2 + (-z_2)|$

$|z_1| \leq |z_1 + z_2| + |-z_2|$

$|z_1| \leq |z_1 + z_2| + |z_2|$

$|z_1| - |z_2| \leq |z_1 + z_2| \longrightarrow (B)$

Now for $z_2 = |z_1 + z_2 - z_1| = |z_1 + z_2 + (-z_1)|$

$|z_2| \leq |z_1 + z_2| + |z_1|$

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$-|z_1| + |z_2| \le |z_1 + z_2|$

$-|z_1 + z_2| \le (|z_1| - |z_2|) \longrightarrow (b)$

on Combining (a) and (b)

$\Rightarrow -|z_1 + z_2| \le |z_1| - |z_2| \le |z_1 + z_2|$

$\forall\ x, a \in \mathbb{R};$

iff $-a \le x \le a \Rightarrow |x| \le a$

$|(|z_1| - |z_2|)| \le |z_1 + z_2| \longrightarrow (B)$

on combing (A) and (B) we get

real inequality.

Example:-

find the Maximum and Minimum value

of $f(x) = \left[\dfrac{2}{3+ie^{ix}}\right] \in \mathbb{R} \ge 0$

Triangular Inequality ; $|z_1| - |z_2| \le |z_1 + z_2|$

$\le |z_1| + |z_2| \longrightarrow (A)$

$\because |3 + ie^{ix}| \le |3| + |ie^{ix}| = 3 + 1 = 4$

$|3 + ie^{ix}| \le 4$

$\dfrac{1}{3+ie^{ix}} \ge 4$

$\dfrac{2}{3+ie^{ix}} > \dfrac{2}{4}$

$\dfrac{|2|}{|3+ie^{ix}|} \ge \dfrac{1}{2}$

$$f(x) = \left|\frac{2}{3+ie^{ix}}\right| \geq \frac{1}{2}$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \geq |z_1|$$

Now for $|3+ie^{ix}| \geq |3| - |ie^{ix}|$

$$= 3 - 1 = 2$$

$$\frac{1}{3+ie^{ix}} \leq \frac{1}{2}$$

Minium value is $\frac{1}{2}$

Max value is 1

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$$4x^2+8x+4+4y^2-1-2y-y^2-x^2=0$$

$$3x^2+3y^2+8x-2y+3=0$$ Answer

lecture # 04:-

Euler's formula, De Moivre's theorem and their applications

1. Euler's formula:-

$$\cos x + i\sin x = e^{ix}$$

let $z = x+iy = (r\cos x + i\sin x) = r\,cis\,x \rightarrow ($

By Maclurian Series:-

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$$

$$i\sin x = ix-\frac{ix^3}{3!}+\frac{ix^5}{5!}$$

$$= ix+i.i^2\frac{x^3}{3!}+i\,i^4\frac{x^5}{5!}+\cdots$$

$$i\sin x = ix+\frac{(ix)^3}{3!}+\frac{(ix^5)}{5!}+\cdots \rightarrow (ii)$$

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots$$

$$\cos x = 1+\frac{(ix)^2}{2!}+\frac{(ix)^4}{4!} \longrightarrow (iii)$$

Put in eq (1)

$$z = r\left[\left\{1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \cdots\right\} + \left\{ix + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \cdots\right\}\right]$$

$$z = r\left[1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots\right]$$

$$\because \left\{ e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \cdots \right.$$

$$z = re^{ix}$$

$$\cancel{r}(\cos x + i\sin x) = \cancel{r}e^{ix}$$

$$\cos x + i\sin x = e^{ix}$$

Put $x = \pi$

$$\cos\pi + i\sin\pi = e^{i\pi}$$

$$-1 + 0 = e^{i\pi}$$

$$\boxed{e^{i\pi} + 1 = 0} \Rightarrow \text{Euler's Identity}$$

Examples of Euler's formula:-

If $z_k = \cos\left(\frac{\pi}{2k}\right) + i\sin\left(\frac{\pi}{2k}\right)$; Then

evaluate $\prod_{k=1}^{\infty} z_k$. $\qquad \prod$ = product

$$z_k = \cos\left(\frac{\pi}{2k}\right) + i\sin\left(\frac{\pi}{2k}\right) = e^{i\left(\frac{\pi}{2k}\right)}$$

$$\prod_{k=1}^{\infty} = z_1 \cdot z_2 \cdot z_3 + \cdots \infty = e^{i\left(\frac{\pi}{2}\right)} \cdot e^{i\left(\frac{\pi}{2^2}\right)} \cdot e^{i\left(\frac{\pi}{2^3}\right)}$$

$k=1$

$= e^{i\left(\frac{\pi}{2}\right) + i\left(\frac{\pi}{2^2}\right) + i\left(\frac{\pi}{2^3}\right) + \cdots \infty}$

$e^{i\frac{\pi}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \infty\right]}$

$\Rightarrow$ { Here $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$ is a geometric Series.

$\Rightarrow a_1 = a = 1$; $r = \frac{a_2}{a_1} = \frac{1/2}{1} = 1/2$

{ $S_\infty = \frac{a}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = 2$

$= e^{i\frac{\pi}{\cancel{2}}(\cancel{2})}$

$\Rightarrow e^{i\pi} = \cos\pi + i\sin\pi$

$-1 + i(0) = -1$

$e^{i\pi} = -1$ Euler's Identity

DeMoivre's Theorem (1- Integers Case)

for any $z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$ for any $n \in \mathbb{Z}^+$

$= e^{in\theta}$

$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$ DeMoivre's Theorem

for $n \in \mathbb{Z}^-$

$(\cos\theta + i\sin\theta)^{-m} = (e^{i\theta})^{-m}$

$e^{-im\theta} = \cos(-m\theta) + i\sin\theta(-m\theta)$

$= \cos m\theta - i\sin m\theta$ $\qquad \because \cos(-\alpha) = \cos\alpha$

$(\cos\theta + i\sin\theta)^{-m} = \cos m\theta - i\sin m\theta$ $\qquad \sin(-\alpha) = \sin\alpha$

*De Moivre Theorem (2 - Rational Case)

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$n = \frac{p}{q} \in Q \; ; \; Q \neq 0$

for $\frac{1}{q} \in q$:-

$(\cos\theta + i\sin\theta)^{1/q} = (e^{i\theta})^{1/q} = e^{i\frac{\theta}{q}}$

$e^{i\theta/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} \rightarrow (1)$

For $\sin\theta = \sin\theta \pm 2k\pi$

$\cos\theta = \cos(\theta \pm 2k\pi)$

$(\cos\theta + i\sin\theta)^{1/q} = \cos\left(\frac{\theta + 2\pi k}{q}\right) + i\sin\left(\frac{\theta + 2k\pi}{q}\right)$

Put $k = 0$

$(\cos\theta + i\sin\theta)^{1/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} \rightarrow (B)$

(B) is one of the value of (A), whenever $k=0$ for $\frac{1}{q} \in Q$

Wolfram Alpha

from (B)

$$\left[(\cos\theta + i\sin)^{1/q}\right]^p = \left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^p$$

$$= \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

- **De Moivres Theorem Example.**

Evalute $(-1+i)^{100}$

let $z = x+iy = -1+i$

$x = -1$, $y = 1$

$r = \sqrt{(-1)^2+(1)^2}$ $\Rightarrow$ $\mathbb{R}(\theta)$ lies in 2nd quad

$r = \sqrt{2}$

$$\operatorname{Arg} z = \theta = \pi - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \pi - \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \pi - \tan^{-1}(1)$$

$$\theta = \pi - \frac{\pi}{4} \Rightarrow \frac{3\pi}{4}$$

Now $z = r(\cos\theta + i\sin\theta) = \sqrt{2}\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$$(-1+i)^{100} = \left[\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)\right]^{100} = (\sqrt{2})^{100} \cdot cis\left[\frac{3\pi}{4}\times 100\right.$$

$$= (2^{1/2})^{100} \cdot cis(3\pi \times 25)$$

$$= 2^{50} \cdot cis(74\pi + \pi)$$

$$= 2^{50} \cdot cis(37(2\pi) + \pi)$$

$$= 2^{50} \cdot cis(\pi)$$

$$= 2^{50} \cos\pi + i\sin\pi$$

$$= 2^{50}(-1) + i(0)$$

$$-2^{50}$$

lecture #05:-

DeMoivers Theorem (Application-1)

To express ~~Con~~ Cosnθ and Sinnθ as finite Sums of powers of Trigonometric function of θ, where n is positive:-

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + (i\sin\theta)^2 + 2\cos\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta + i(2\cos\theta\sin\theta)$

By De Moiver's law:-

$\cos 2\theta + i\sin 2\theta = \cos^2\theta - \sin\theta + i(2\cos\theta\sin\theta)$

Real:-

$\cos 2\theta = \cancel{\cos^2}\cos^2\theta - \sin^2\theta$

$i\sin 2\theta = 2\cos\theta\sin\theta$

Bionomial cofficent:-

$(a+b)^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^{2-0}b^0 + \binom{2}{1}a^{2-1}.b + \binom{2}{2}a^{2-2}.b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^2$

$(a+b)^n = \binom{n}{0}a^{n-0}b^0 + \binom{n}{1}a^{n-1}.b^1 + \binom{n}{2}a^{n-2}.b + \cdots \binom{n}{r}a^{n-r}.b^r + \binom{n}{r+1}a^{n-(r+1)}b^{r+1} + \cdots + \binom{n}{n-1}a^{n-(n-1)}b^{n-1} + \binom{n}{n}a^{n-n}b^n$

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② De'Moiver's Theorem (Application-1 examples)

find the trigonometric Identities for $\sin 5\theta$, $\cos 5\theta$, $\tan 5\theta$.

let $\sin\theta = s$, $\cos\theta = c$, $\tan\theta = t$

By De'Moiver's theorem:-

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$= (c + is)^5$

$= \binom{5}{0} c^{5-0}(is)^0 + \binom{5}{1} c^{5-1}(is)^1 + \binom{5}{2} c^{5-2}(is)^2$

$+ \binom{5}{3} c^{5-3}(is)^3 + \binom{5}{4} c^{5-4}(is)^4 + \binom{5}{5} c^{5-5}$

$(is)^5$

$= \frac{5!}{(5-0)!\,0!} c^5 \cdot 1 + \frac{5!}{(5-1)!\,1!} c^5(is) + \frac{5!}{(5-2)!\,2!} c^5 (-1)s^2$

$+ \frac{5!}{(5-3!)\,3!} c^2 (-1) s^3 + \frac{5!}{(5-4)!\,4!} c \cdot 1 \cdot s^4 + \frac{5!}{(5-5)!\,5!} \cdot 1 \cdot i s^5$

$= c^5 + 5c^4(is) + 10c^3(-s^2) + 10c^2(-is^3) + 5cs^4$
is^4

$\cos 5\theta + i\sin 5\theta = (c^5 - 10c^3s^2 + 5cs^4) + i(s^5 - 10c^2s^3$
$+ 5c^4 s)$

equating Real and imaginary parts.

$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4 \longrightarrow$ (i)

$\sin 5\theta = s^5 - 10c^2 s^3 + 5c^4 s \longrightarrow$ (ii)

$\sin\theta = 2\sin\theta\cos\theta$

$\cos^2\theta + \sin\theta = 1$

$C^2 = 1 - S^2$

$S^2 = 1 - C^2$

$S^4 = (S^2)^2$

from equation (1)

$\cos 5\theta = C^5 - 10C^3S^2 + 5CS^4$

$= C^5 - 10C^3(1-C^2) + 5C(1-C^2)^2$

$= C^5 - 10C^3 + 10C^5 + 5C(1 + C^4 - 2C^2)$

$= C^5 - 10C^3 + 10C^5 + 5C + 5C^5 - 10C^3$

$\cos 5\theta = 16C^5 - 20C^3 + 5C$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

from equation (2)

$\sin 5\theta = S^5 - 10C^2S^3 + 5C^4S$

$= S^5 - 10(1-S^2)S^3 + 5S(1-S^2)^2$

$= S^5 - 10S^3 + 10S^5 + 5S(1 + S^4 - 2S^2)$

$= S^5 - 10S^3 + 10S^5 + 5S + 5S^5 - 10S^2$

$= 16S^5 - 20S^3 + 5C$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$\frac{\sin 5\theta}{\cos 5\theta} = \frac{s^5 - 10c^2s^3 + 5c^4s}{c^5 - 10c^3s^2 + 5c \cdot s^4}$$

Dividing Numetor and Denometar by c^5

$$\tan 5\theta = \frac{s^5/c^5 - \frac{10c^2s^3}{c^5} + 5c^4s/c^5}{\frac{c^5}{c^5} - \frac{10c^3s^2}{c^5} + \frac{5cs^4}{c^5}}$$

$$= \frac{t^5 - 10t^3 + 5t}{1 - 10t^2 + 5t^4}$$

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

De' Moivers Theorem Application '2'

Express $\cos^{-1}(\theta)$ and $\sin^{-1}(\theta)$ as sum of Sines and Cosines of Mutiples of θ

let

$x = \cos\theta + i\sin\theta \rightarrow (I)$

$$\frac{1}{x} = \frac{1}{\cos + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$$

$$= +\cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta \rightarrow (2)$$

By (1) + (2)

$x + \frac{1}{x} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$\boxed{x + \frac{1}{x} = 2\cos\theta}$

By ① - ②

$x - \frac{1}{x} = \cos\theta + i\sin\theta - \cos\theta - i\sin\theta$

$\boxed{x - \frac{1}{x} = +2i\sin\theta}$

for n^{th} powers

$x^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta \rightarrow ③$

$\frac{1}{x^n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta - i\sin n\theta \rightarrow ④$

By ③ + ④

$\boxed{x^n + \frac{1}{x^n} = 2\cos n\theta}$

By ③ - ④

$\boxed{x^n - \frac{1}{x^n} = 2i\sin n\theta}$

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Example :-

$\cos^4\theta \cdot \sin^3\theta$ as sum of sines and cosines of multiples of θ

$(2\cos\theta)^4 (2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4 \left(x-\frac{1}{x}\right)^3$

$2^4 \cdot \cos^4\theta \cdot 2^3 i^3 \sin^3\theta = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$

$$= \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3$$

$\Rightarrow -2^7 i \cos^4\theta \cdot \sin^3\theta = \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^3$

$\because (a-b)^3 = a^3 - b^3 - 3ab(a-b)$

$= \left(x+\frac{1}{x}\right)\left[(x^2)^3 - \left(\frac{1}{x^2}\right)^3 - 3x^2\frac{1}{x^2}\left(x^2-\frac{1}{x^2}\right)\right]$

$= \left(x+\frac{1}{x}\right)\left(x^6 - \frac{1}{x^6} - 3x^2 + \frac{3}{x^2}\right)$

$= x^7 - \frac{1}{x^5} - 3x^3 + \frac{3}{x} + x^5 - \frac{1}{x^7} - 3x + \frac{3}{x^3}$

$= \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$

$= 2i\sin7\theta + 2i\sin5\theta - 3i\sin3\theta - 3(2i\sin\theta)$

$-2^7 i\cos^4\theta\sin^3\theta = 2i\sin7\theta + 2i\sin5\theta - 3i\sin3\theta - 3(2i\sin\theta)$

By $\div$ing i and on bothsides.

$\cos^4\theta \cdot \sin^3\theta = \frac{-2}{2^7}\left[\sin7\theta + \sin5\theta - 3\sin3\theta - 3\sin\theta\right]$

$$\cos^4\theta \sin^3\theta = \frac{-1}{64}\left(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right)$$

Example:-

if $x + \frac{1}{x} = 2\cos\theta,\ y + \frac{1}{y} = 2\cos\phi;$

Then $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

$m, n \in \mathbb{Z}$.

$\because\ x + \frac{1}{x} = 2\cos\theta$

$$\frac{x^2+1}{x} = 2\cos\theta$$

$$x^2+1 = 2\cos\theta\, x$$

$$x^2 - 2\cos\theta\, x + 1 = 0$$

By Quardatic formulas-

$$x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2\sqrt{(-1)(1-\cos^2\theta)}}{2}$$

$$= \frac{\cancel{2}\left(\cos\theta \pm \sqrt{-\sin^2\theta}\right)}{\cancel{2}}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$x = \cos\theta \pm i\sin\theta$$

So taking $x = \cos\theta + i\sin\theta$
$= cis\theta = e^{i\theta}$

$x^m = (e^{i\theta})^m = e^{im\theta} = \cos(m\theta + i\sin(m\theta)$
$= cis\, m\theta$

$x^{-m} = (e^{i\theta})^{-m} = e^{i(-m\theta)} = \cos(-m\theta) + i\sin(-m\theta$

Similarly:-
$y^m = cis\, m\phi$ & $y^{-m} = cis(-m\phi)$

Now

$$x^m y^n + \frac{1}{x^m y^n} = cis\, m\theta \cdot cis\, n\phi + cis(-m\theta) \cdot cis(-n\phi)$$

$= cis(m\theta + n\phi) + cis(-m\theta - n\phi)$
$= [\cos(m\theta + n\theta) + i\sin(m\theta + n\theta)] + [\cos(-m\theta - n\theta) + i\sin(-m\theta - n\phi)]$

$= \cos(m\theta + n\phi) + \cancel{i\sin(m\theta + n\phi)}$
$+ \cos(m\theta + n\phi) - \cancel{i\sin(m\theta + n\phi)}$

$= 2\cos(m\theta - n\phi)$

Lecture # 06:-

Roots of a Complex Number

1- Roots of an equation (Introduction)

$f(x)=0 \quad , \quad y=f(x)=0$

Real root وہ ویلیو ہوتی ہے جہاں x-axis پر Meet کرتا ہے وہ (انٹرسیکٹ)

(2) Square and cube root of unity.

let one root of unity

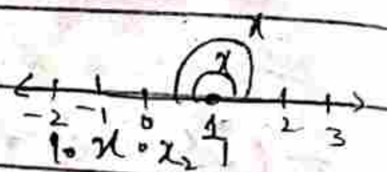
① $x=1$

② $x^2=1$

$x^2-1=0$

$(x-1)(x+1)=0$

$x=\pm 1$



③ cube roots:-

$x^3=1$

$\Rightarrow x^3-1=0$

$(x)^3-(1)^3=0$

$\because a^3-b^3=(a-b)(a^2+ab+b^2)$

$(x-1)(x^2+x+1)=0$

$x=1 \quad , \quad x^2+x+1=0$ i.e quad eq

$$x=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$$

$$x=\frac{-1\pm\sqrt{-3}}{2}$$

$$\omega=x_1=\frac{-1+\sqrt{-3}}{2} \quad , \quad x_2=\frac{-1-\sqrt{-3}}{2}$$

$$\omega = \frac{-1+\sqrt{3}i}{2} \quad ; \quad \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

$$\omega^2 = \left(\frac{-1+\sqrt{-3}}{2}\right)^2$$

$$\omega^2 = \frac{(-1)^2 + (\sqrt{3})^2 + (2)(-1)(\sqrt{-3})}{2}$$

$$= \frac{-1-\sqrt{-3}}{2}$$

$1, \omega, \omega^2$ are cube roots of unity

Observations :-

(1) Sum ; $1+\omega+\omega^2 = 0$

(ii) product; $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$

3. nth roots of unity.-

$x^n = 1$

$x^n - 1 = 0$

$\Rightarrow (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots x^2 + x + 1) = 0$

$(x-1) = 0, \quad \boxed{x = 1}$

$(x^{n-1} + x^{n-2} + x^{n-3} + \cdots x^2 + x + 1) = 0$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x^3 - 1 = 0$

$(x-1)(x^2+x+1) =$

$x^4 - 1 = 0$

$(x-1)(x^3+x^2+x+1$

$x^n = 1 = 1 + i(0) = \cos(0) + i(\sin 0)$

$x^n = \operatorname{cis} 0$

$\sqrt[n]{x^n} = \sqrt[n]{\operatorname{cis} 0}$

$(x^n)^{1/n} = (\operatorname{cis} 0)^{1/n} = (\operatorname{cis}(0+2k\pi))^{1/n}$

~~$x = \operatorname{cis} \frac{1}{n}$~~

$$x = Cis\left(\frac{2k\pi}{n}\right) \longrightarrow (i) \quad k \in \mathbb{Z}$$

for $K = 0$;

$$x_0 = Cis(0) = \cos(0) + i\sin 0 = 1$$

$K = 1$;

$$x_1 = Cis\left(\frac{2(1)\pi}{n}\right) = Cis\left(\frac{2\pi}{n}\right) = \omega$$

$K = 2$

$$x_2 = Cis\left(\frac{2(2)\pi}{n}\right) = Cis\left(\frac{4\pi}{n}\right) = \omega^2$$

$\vdots$

$K = n-1$

$$x_{n-1} = Cis\left(\frac{2(n-1)\pi}{n}\right) = \omega^n$$

$K = n$,

$$x_n = Cis\left(\frac{2n\pi}{n}\right)$$

$$= Cis 2\pi = \cos(2\pi) + i\sin(2\pi)$$

$$= 1 + i(0) = 1$$

$1, \omega, \omega^2, — \omega^{n-1}$ are nth roots of unity.

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5. nth roots of any Real Number:-

$a \in \mathbb{R}$

$x^3 = a$

$x^3 - a = 0$

$x^3 - (a^{1/3})^3 = 0$

$x^3 - (a^{1/3})^3 = 0$

$\Rightarrow (x - a^{1/3})(x^2 + a^{1/3}x + a^{2/3}) = 0$

$x - a^{1/3} = 0$; $x^2 + a^{1/3}x + a^{2/3} = 0$

$x = a^{1/3}$, $x = \dfrac{-a^{1/3} \pm \sqrt{a^{2/3} - 4(a^{2/3})}}{2 \cdot 1}$

$x = \dfrac{-a^{1/3} \pm \sqrt{(a^{1/3})^2 (1-4)}}{2}$

$= \dfrac{-a^{1/3} \pm a^{1/3}\sqrt{-3}}{2}$

$= a^{1/3}\left[\dfrac{-1 \pm \sqrt{-3}}{2}\right]$

$= a^{1/3}(\omega, \omega^2)$

$x = a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$

for nth roots of '1'

$1, \omega, \omega^2 \text{ --- } \omega^{n-1}$

for $b \in \mathbb{R}$

$b^{1/n}, b^{1/n}\omega, b^{1/n}\omega^2 \ldots\ldots b^{1/n}\omega^{n-1}$

6. Roots of any Complex number (Example):-

Find the 4th roots of $-2\sqrt{3}+2i$.

let $\omega^4 = -2\sqrt{3}+2i = x+iy$ ---(1)

$x = -2\sqrt{3}$, $y = 2$

T. ray of ~~arg~~ argu $= \theta$ lies in 2nd quad

$r = \sqrt{x^2+y^2}$

$r = \sqrt{(-2\sqrt{3})^2 + 2^2}$

$= \sqrt{16} = 4$

Argument of z -

$\text{Arg } z = \pi - \tan^{-1}\left(\frac{y}{x}\right)$

$= \pi - \tan^{-1}\left(\frac{2}{+2\sqrt{3}}\right)$

$= \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \pi - \pi/6$

$= 5\pi/6$

$\omega^4 = x+iy = r\,cis\theta = 4\,cis\theta$

$4\,cis\left(\frac{5\pi}{6}\right)$

$\Rightarrow \omega^4 = 4\,cis\left(\frac{5\pi}{6} + 2k\pi\right)$; $k \in \mathbb{Z}$

$= 4\,cis\left(\frac{5\pi + 12k\pi}{6}\right)$

$$w^{4 \times \frac{1}{4}} = 4\,cis\left\{\left(\frac{(5+12k)\pi}{6}\right)\right\}^{1/4}$$

$$w = 4^{1/4} \cdot \left[cis\left(\frac{(5+12k)\pi}{6}\right)\right]^{1/4}$$

$$w = \sqrt{2} \cdot cis\frac{1}{4}\left(\frac{(5+12k)\pi}{6}\right)$$

$$= \sqrt{2}\, cis\left(\frac{(5+12k)\pi}{24}\right) \longrightarrow (1)$$

for $k = 0$

$$w_0 = \sqrt{2}\, cis\left(\frac{5+12(0)\pi}{24}\right)$$

$$= \sqrt{2}\, cis\, \frac{5\pi}{24}$$

for $k = 1$

$$w_1 = \sqrt{2}\, cis\left(\frac{5\pi + 24\pi}{24}\right)$$

$$= \sqrt{2}\, cis\left(\frac{29\pi}{24}\right)$$

for $k = 2$;

$$w_2 = \sqrt{2}\, cis\left(\frac{5\pi + 36\pi}{24}\right)$$

$$= \sqrt{2}\, cis\left(\left(\frac{41\pi}{24}\right) - 2\pi\right) = \sqrt{2}\, cis\left(\frac{-7\pi}{24}\right)$$

$k = 3$;

$$w_3 = \sqrt{3}\, cis\left(\frac{53\pi}{24}\right)$$

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lecture #07:-

Basic Elementry Functions:-

(1) Agebric function => polynomial

(2) Exponential and lograthmic function

(3) Trigonometic function and their inverse.

Exponential function:-

for any $y \in \mathbb{R}$; Euler's formula;

$$e^{iy} = \cos y + i\sin y \rightarrow (1)$$

if $z = (x+iy) \in \mathbb{C}$

from (1)

xing both sides by $\langle e^x \rangle$

$$e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

$$\boxed{e^z = e^x(\cos y + i\sin y)} \rightarrow (A)$$

i.e defined to be an exponential function for $z \in \mathbb{C}$.

Case (I)

z is pure real so

$z_{Img} = 0 = y$

$(A) \Rightarrow e^{z} = e^{x}(\cos 0 + i \sin 0)$

$= e^{x}(1 + i0)$

$e^{z} = e^{x}$

$\boxed{e^{i\pi} = -1}$

Case (ii)

z is pure imaginary $\Rightarrow Re\, z = 0 = x$

$(A) \Rightarrow e^{z} = e^{0}(\cos y + i \sin y) = 1 \cdot e^{iy}$

$e^{z} = e^{iy}$

Example:-

for any $z = (x+iy) \in \mathbb{C}$; $e^{z} = 0$??

if $e^{z} = 0$, $e^{x}(\cos y + i \sin y) = 0$

$\Rightarrow \cos y + i \sin y = 0$; $e^{x} \neq 0 \ \forall x \in \mathbb{R}$

$\cos y + i \sin y = 0 + i0$

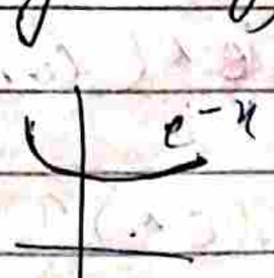
Comparing real and Img

$\cos y = 0$, $\sin y = 0$ $y \in \mathbb{R}$.

There does not exist $y \in \mathbb{R}$ so that above holds.

$(\cos y + i \sin y) \neq 0 \Rightarrow e^{x}(\cos y + i \sin y)$

$e^{z} \neq 0$



Period of $(z) e^{z}$

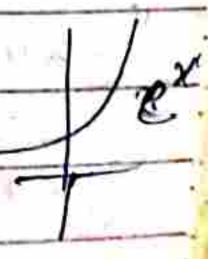
Since $e^{z} = e^{x}(\cos y + i \sin y)$

for any $z = (x+iy) \in \mathbb{C}$

$e^{z} = e^{x}\{\cos(y + 2k\pi) + i \sin(y + 2k\pi)\}$

$= e^{x} e^{i(y + 2k\pi)}$

$= e^{x + iy + 2k\pi}$



$= e^{x+iy+i2k\pi}$

$e^{z} = e^{z+2k\pi i}$

e^{z} is periodic with period $2k\pi i$
$k \in \mathbb{Z}$.

Properties of Complex Numbers:

i) $e^{z_1+z_2} = e^{z_1} e^{z_2}$

ii) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

iii) $(e^{z})^{n} = e^{nz}$; $n \in \mathbb{Z}$.

$\Downarrow$

Taking $(e^{z})^{n}$
for any $z = (x+iy) \in \mathbb{C}$; $n \in \mathbb{Z}$

$(e^{z})^{n} = (e^{x+iy})^{n} = (e^{x} \cdot e^{iy})^{n}$

$= [e^{x}(\cos y + i\sin y)]^{n}$

$= (e^{x})^{n} \cdot (\cos y + i\sin y)^{n}$

$= e^{xn}(\cos ny + i\sin ny)$

$= e^{nx} \cdot e^{iny} = e^{nx+iny}$

$= e^{n(x+iy)} = e^{nz}$ R.H.S

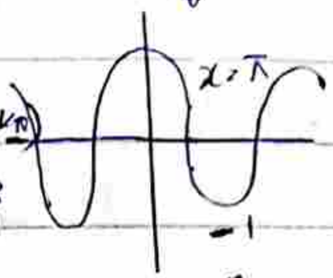
Solve the equation :-

$$e^z = -1 \; ; \; z \in \mathbb{C}$$

Solution :-

$$e^z = -1 = -1 + i(0)$$

$\cos(\pi) = -1$

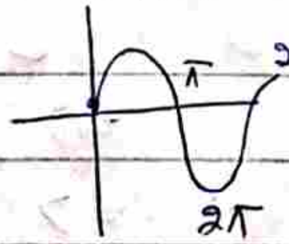


$$e^z = \cos\pi + i\sin\pi$$

$$e^z = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$$

$$e^z = \cos\{(2k+1)\pi\} + i\sin\{(2k+1)\pi\}$$

sin x



$$e^z = e^{(2k+1)\pi i}$$

$$e^{z_1} = e^{z_2} \neq \cancel{2k\pi} \; z_1 = z_2$$

But $z_1 = z_2 + 2k(i\pi)$

$$z_1 = (2k+1)\pi i + 2k'(i\pi)$$
$$= 2k\pi i + \pi i + 2k' i\pi$$
$$= (2k + 1 + 2k')\, i\pi$$
$$= [2(k + k') + 1]$$
$$= i(2(k'') + 1)$$
$$= i\,(2(k'' + 1)]\,\pi$$

where $k + k' = k''$.

$$k'' \in \mathbb{Z}$$

Prove that $|e^{-2z}| < 1 \iff Re(z) > 0$; $z \in \mathbb{C}$.

Let :-

$p : |e^{-2z}| < 1$,

& $q : Re\ z > 0$, $z \in \mathbb{C}$

$P \to q$:-

P is given & to prove q

$P \Rightarrow |e^{-2z}| < 1$ if $z = (x+iy) \in \mathbb{C}$

$\Rightarrow |e^{-2(x+iy)}| < 1$

$\Rightarrow |e^{-2x-2iy}| < 1$

$\Rightarrow |e^{-2x} \cdot e^{-2iy}| < 1$

$\Rightarrow |e^{-2x}| \cdot |e^{-2iy}| < 1$

$\Rightarrow |e^{-2x}| \cdot |e^{i(-2y)}| < 1$

$\Rightarrow |e^{-2x}| \, | \cos(-2y) + i \sin(-2y) | < 1$

$\Rightarrow |e^{-2x}| \, | \cos^2(-2y) + \sin^2(-2y) | < 1$

$\Rightarrow |e^{-2x}| < 1$

$\Rightarrow e^{-2x} < 1 \quad \because e^{-2x} > 0 \ \forall x \in \mathbb{R}$

$\Rightarrow \frac{1}{e^{2x}} < 1 \Rightarrow 1 < e^{2x} \Rightarrow e^{2x} > 1]$

$\Rightarrow e^{2x} > e^{0}$

e^{2x} is $\uparrow$ function

$\Rightarrow 2x > 0 \Rightarrow x > 0 \Rightarrow \boxed{Re\,z > 0}$

Now conversly we have given 'q' & we prove 'p'

$q \Rightarrow Re\,z > 0 \Rightarrow x > 0$

$\therefore |e^{-2z}| = |e^{-2(x+iy)}|$

$= |e^{-2x-2iy}|$

$= |e^{-2x} . e^{-2iy}|$

$|e^{-2z}| = e^{-2x} < 1 \quad , x > 0$

$Re\,z > 0$

Question :-

Examine the behavior of

e^{x+iy} as $x \to -\infty$

e^{x+iy} as $y \to +\infty$

Case:- $e^{x+iy} \quad x \to -\infty$

$\lim_{x \to -\infty} e^{x} . e^{iy}$

$= (\lim_{x \to +\infty} e^{x}) e^{iy}$

$= (0) e^{iy} = 0$

$e^{x} \to 0$, whenever $x \to -\infty$

Case:- $\lim_{y \to \infty} e^{z} = \lim_{y \to \infty} e^{x+iy}$

$= \lim_{y \to \infty} e^{x} . e^{iy} \Rightarrow e^{x} (\lim_{y \to \infty} e^{iy})$

$$= e^{x} \lim_{y \to \infty} (\cos y + i \sin y)$$

Applying limit :-

$$= e^{x} [\underbrace{\cos\infty + i(\infty)}_{\text{Circular motion.}}]$$

$\cancel{= e^{x}}$

e^{x} rotates over & over again about origion having radius. e^{x}

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lecture # 8

Trigonometric Function

from Euler's $\Rightarrow$ $e^{iy} = \cos y + i\sin y \rightarrow ①$, $\forall y \in \mathbb{R}$

$e^{-iy} = e^{i(-y)} = \cos(-y) + i\sin(-y)$

$e^{-iy} = \cos y - i\sin y \rightarrow ②$

By ① + ②

$e^{iy} + e^{-iy} = (\cos y + i\sin y) + (\cos y - i\sin y)$

$e^{iy} + e^{-iy} = 2\cos y$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

By ① − ②

$e^{iy} - e^{-iy} = (\cos y + i\sin y) - (\cos y - i\sin y)$

$e^{iy} - e^{-iy} = 2i\sin y$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \forall y \in \mathbb{R}$$

on the same analogy, we can define for $z \in \mathbb{C}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \rightarrow ③ \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \rightarrow ④$$

$$\Rightarrow \sec z = \frac{2}{e^{iz} + e^{-iz}}, \qquad \operatorname{cosec} z = \frac{2i}{e^{iz} - e^{-iz}}$$

By ③ ÷ ④

$$\frac{\cos z}{\sin z} = \frac{e^{iz} + e^{-iz}/2}{e^{iz} - e^{-iz}/2i} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$\cot z$

$$\tan z = \frac{1}{\cot z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

* Extended Euler's formula :-

$$e^{iz} = \cos z + i\sin z \; ; \; z \in \mathbb{C}$$

$$R.H.S = \cos z + i \sin z$$

$$= \frac{e^{iz} + e^{-iz}}{2} + \cancel{i}\left(\frac{e^{iz} - e^{-iz}}{2\cancel{i}}\right)$$

$$= \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2}$$

$$= \frac{e^{iz} \cancel{+ e^{-iz}} + e^{iz} \cancel{- e^{-iz}}}{2}$$

$$= \frac{\cancel{2} e^{iz}}{\cancel{2}} = e^{iz} \; ; \; z \in \mathbb{C}$$

$$e^{iz} = \cos z + i\sin z \; ; \; z \in \mathbb{C}$$

* Complex Tringonometric functions:-

prove that:-

(i) $\sin(z+2\pi) = \sin z$

(ii) $\cos(z+2\pi) = \cos z$

(iii) $\tan(\pi + z) = -\tan z$

(iv) $\cos(z+\pi) = -\cos z$

(v) $\sin(z+\pi) = -\sin z, \quad z \in \mathbb{C}$

$$\downarrow$$

$\sin(z+\pi) = -\sin z.$

$$L.H.S = \sin(z+\pi) = \frac{e^{i(z+\pi)} - e^{-i(z+\pi)}}{2i}$$

$$= \frac{e^{iz+i\pi} - e^{-iz-i\pi}}{2}$$

$$= \frac{e^{iz}.e^{i\pi} - e^{-iz}.e^{-i\pi}}{2i}$$

$$= \frac{e^{iz}(\cos\pi + i\sin\pi) - e^{-iz}(\cos(-\pi) + i\sin(-\pi))}{2i}$$

$$= e^{iz}(-1 + i(0)) - e^{-iz}(-1 + i(0)) \Big/ 2i$$

$$= \frac{-e^{iz} + e^{-iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z$$

Prove that (i) $\sin^2 z + \cos^2 z = 1$

(ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(iii) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1$
$= 1 - 2\sin^2 z$

Middle 1 $= \cos^2 z - \sin^2 z$

$$= (\cos z)^2 - (\sin z)^2$$

$$= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \frac{(e^{iz})^2 + (e^{-iz})^2 + 2e^{iz} \cdot e^{-iz}}{4} - \left[\frac{(e^{iz})^2 + (e^{-iz})^2 - 2e^{iz} \cdot e^{-iz}}{4i^2}\right]$$

$$= \frac{e^{2iz} + e^{-2iz} + 2e^{iz - iz}}{4} - \left[\frac{e^{2iz} + e^{-2iz} - 2e^{iz} \cdot e^{-iz}}{4(-1)}\right]$$

$$= \frac{e^{2iz} + e^{-2iz} + 2}{4} + \frac{e^{2iz} + e^{-2iz} - 2}{4}$$

$$= \frac{e^{2iz} \cancel{+ e^{-2iz}} \cancel{+2} - e^{2iz} \cancel{+ e^{-2iz}} \cancel{-2}}{4}$$

$$= \frac{2e^{2iz} - 2e^{-2iz}}{4}$$

$$\frac{e^{i2z} + e^{-i(2z)}}{e} = \cos 2z$$

Middle : 2:-

$$2\cos^2 x - 1 = 2(\cos x)^2 - 1$$

$$= 2\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 - 1$$

$$= \cancel{2}\left[\frac{e^{2ix} + e^{-2ix} + 2e^{ix}e^{-ix}}{\cancel{4}\,2}\right] - 1$$

$$= \frac{e^{2ix} + e^{-2ix} + 2}{2} - 1$$

$$= \frac{e^{2ix} + e^{-2ix} + 2 - 2}{2}$$

$$= \frac{e^{2ix} + e^{-2ix}}{2} = \cos 2x = L.H.S$$

$$R.H.S = 1 - 2\sin^2 x = 1 - 2(\sin x)^2$$

$$= 1 - 2\left[\frac{e^{ix} - e^{-ix}}{2i}\right]^2$$

$$= 1 - \cancel{2}\left[\frac{e^{2ix} + e^{-2ix} - 2e^{ix}e^{-ix}}{4i^2}\right]$$

$$= 1 - \cancel{2}\left[\frac{e^{2ix} + e^{-2ix} - 2}{2}\right]$$

$$= 1 + \frac{e^{2ix} + e^{-2ix} - 2}{2}$$

$$= \frac{\cancel{2} + e^{2ix} + e^{-2ix} - \cancel{2}}{2}$$

$$= \frac{e^{2ix} + e^{-2ix}}{2} = \cos 2x = L.H.S$$

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Show that $\cos x = 0 \iff x = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$

let:-

$P: \cos x = 0$ & $q: x = (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}$

$P \to q$

given $\cos x = 0$ & to prove $x = (2n+1)\frac{\pi}{2}$

$$P \Rightarrow \cos x = 0 \Rightarrow \frac{e^{ix} + e^{-ix}}{2} \Rightarrow e^{ix} + e^{-ix} = 0$$

$$\Rightarrow e^{ix} + \frac{1}{e^{ix}} = 0 \quad \Rightarrow \frac{(e^{ix})^2 + 1}{e^{ix}} = 0$$

$$e^{2ix} + 1 = 0$$

$$e^{i(2x)} = -1$$

$$-1 + i(0) = \cos\pi + i\sin\pi$$

$$e^{i(2\pi)} = \cos(\pi + n(2\pi)) + i(\sin(\pi + n(2\pi$$

$$= \cos((2n+1)\pi) + i\sin[(2n+1)\pi]$$

$$e^{i(2\pi)} = e^{i(2n+1)\pi}$$

$$i2x = i(2n+1)\pi + 2ki\pi$$

$$2x = \pi(2n+1+2k)$$

$$= \pi(2(n+k)+1)$$

$$2x = (2n'+1)\pi$$

$$x = (2n'+1)\pi/2$$

Lecture # 09

Hyperbolic Function :-

Introduction:-

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

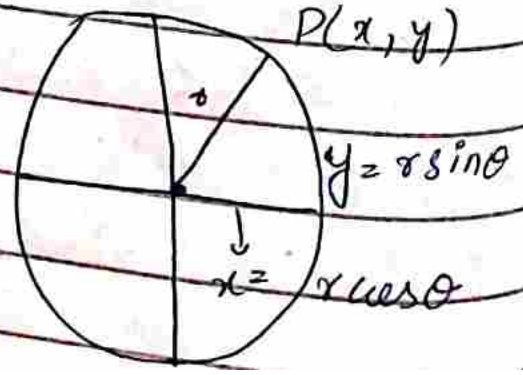
$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}$

In Similar manner, for $z \in \mathbb{C}$;

$\cosh z = \frac{e^z + e^{-z}}{2}$, $\sinh z = \frac{e^z - e^{-z}}{2}$

$\operatorname{sech} z = \frac{2}{e^z + e^{-z}}$, $\operatorname{cosech} = \frac{2}{e^z - e^{-z}}$

$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$; $\coth z = \frac{e^z + e^{-z}}{e^z - e^{-z}}$



2. Relation between Hyperbolic and Circular function

$\forall\ z \in \mathbb{C}$;

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i}$$

$$= e^{i^2(z)} - e^{-i^2(z)} / 2i$$

$$\sin(iz) = \frac{e^{-z} - e^{-(-1)(z)}}{2i}$$

$$\sin(iz) = \frac{e^{-z} - e^{z}}{2i}$$

$$= i\,\frac{e^{-z} - e^{z}}{2i^2}$$

$$= -\frac{i(e^{z} - e^{-z})}{2(-1)}$$

$$\sin(iz) = i\left(\frac{e^{z} - e^{-z}}{2}\right) = i\sinh z$$

$$\boxed{\sin(iz) = i\sinh z}$$

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{-z} + e^{+z}}{2}$$

$$= e^{z} + e^{-z} / 2$$

$$\boxed{\cos(iz) = \cosh z}$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i\sinh z}{\cosh z} = i\,(\ldots$$

$$\boxed{\tan(iz) = i\tanh z}$$

Hyperbolic Identities :-

(i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\operatorname{sech}^2 z + \tanh^2 z = 1$

(iii) $\coth^2 z = \operatorname{cosech}^2 z = 1$

$\cosh^2 z - \sinh^2 z = 1$

$= (\cosh z)^2 - (\sinh z)^2$

$$= \left(\frac{e^{z} + e^{-z}}{2}\right)^2 - \left(\frac{e^{z} - e^{-z}}{2}\right)^2$$

$$= \frac{e^{2z} + e^{-2z} + 2e^{z}\cdot e^{-z}}{4} - \left[\frac{e^{2z} + e^{-2z} - 2e^{z}e^{-z}}{4}\right]$$

$$= \frac{e^{2z} + e^{-2z} + 2e^{0}}{4} - \left[\frac{e^{2z} + e^{-2z} - 2e^{0}}{4}\right]$$

$$\frac{e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2}{4}$$

$$= \frac{4}{4} = 1$$

Hyperbolic function (Application-I)

If $z = (x+iy) \in \mathbb{C}$

Then $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

$\cos z = \cos(x+iy)$

$\because \cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$\cos z = \cos x \cdot \cos(iy) - \sin x \cdot \sin(iy)$

$\because \cos(iy) = \cosh y$

$\because \sin(iy) = i\sinh y$

$\cos z = \cos x \cosh y - \sin x \, i\sinh y$

$= \cos x \cosh y + i(-\sin x \sinh y)$

$|\cos z|^2 = (\cos x \cosh y)^2 + (-\sin x \cdot \sinh y)^2$

$= \cos^2 x \cosh^2 y + \sin^2 x \cdot \sinh^2 y$

$\Rightarrow \because \sin^2 x + \cos^2 x = 1$

$\sin^2 x = (1 - \cos^2 x)$

$\cosh^2 y - \sinh^2 y = 1$

$\cosh^2 y = 1 - \sinh^2 y$

$= \cos^2 x \cdot (1 + \sinh^2 y) + (1 - \cos^2 x) \cdot \sinh^2 y$

$= \cos^2 x + \cancel{\cos^2 x \sinh^2 y} + \sinh^2 y - \cancel{\sinh^2 y \cos^2 x}$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

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(i) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$L.H.S = |\sin z|^2$

$\sin z = \sin(x+iy)$

$\Rightarrow \because \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(x+iy) = \sin x \sin(iy) + \cos x \cos(iy)$

$= \sin x\, i \sinh y + \cos x \cosh y$

$= i(\sin x \sinh y) + \cos x \cosh y$

$|\sin z|^2 \Rightarrow (\sin x \sinh y)^2 + (\cos x \cosh y)^2$

$= \sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y$

$\Rightarrow \sin^2 x(1-\cosh^2 y) + \cos^2 x(1-\sinh^2 y)$

$= \sin^2 x - \sin^2 x \cosh^2 x + \cos^2 x - \cos^2 x \sinh^2 y$

$= \sin^2 x(1-\cosh^2 x) + \cos^2 x(1-\sinh^2 y)$

$= (\sin^2 x + \cos^2 x) + (1-\cosh^2 x)$

$= 1 + \sinh^2 y$

for any $z = (x+iy) \in \mathbb{C}$; Show that

$\sin(i\bar{z}) = \overline{\sin(iz)} \iff$

$x = n\pi$; $n \in \mathbb{Z}$

Solution :-

let:- $P : \sin(i\bar{z}) = \overline{\sin(iz)}$

$q : x = n\pi i$, $n \in \mathbb{Z}$

(i) $P \to q$: 'P' ~~prove~~ given to prove $x = n\pi i$;

$P \Rightarrow \sin(i\bar{z}) = \overline{\sin(iz)}$

$\sin(i(\overline{x+iy})) = \overline{\sin(i(x+iy))}$

$\sin(i(x-iy)) = \overline{\sin(xi+i^2y)}$

$\sin(ix-i^2y) = \overline{\sin(ix-y)}$

$\sin(ix+y) = \overline{\sin(-y+ix)}$

$\because \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin ix\cos y + \cos ix\sin y = \overline{\sin(-y)\cos(ix) + \cos(-y)\sin(ix)}$

$i\sinh x\cos y + \cosh x\sin y = \overline{-\sin y\cosh x + \cos y\, i\sinh x}$

$\cosh x\sin y + i\sinh x\cos y = -\sin y\cosh x + i\sinh x\cos y$

$\cosh x\sin y + i\sinh x\cos y = -\sin y\cosh x - i\sinh x\cos y$

$2\cosh x\sin y + 2i\sinh x\cos y = 0 + i0$

$\div$ing by '2'

$\cosh x\sin y + i\sinh x\cos y = 0 + i0$

Equating Real and im. parts.

$\cosh x\sin y = 0$; $\sinh x\cos y = 0$

$\sinh x \cdot \cos y = 0$

$\sin y = 0$; $\cosh x \neq 0$	If $\sinh x = 0$
$y = \sin^{-1}(0)$	$\frac{e^x - e^{-x}}{2} = 0$
$y = n\pi$, $n \in \mathbb{Z}$	$\Rightarrow e^x - e^{-x} = 0$
	$e^x - \frac{1}{e^x} = 0$
Therefore	$\frac{(e^x)^2 - 1}{e^x}$
$z = x + iy$	
$= 0 + i(n\pi)$	$e^{2x} - 1 = 0$
$z = i(n\pi)$	$e^{2x} = 1 = e^0$
	$2x = 0$, $\boxed{x = 0}$

Conversely 'q' is given & we prove 'p'

$(q \rightarrow p)$

$q \Rightarrow \quad z = in\pi \Rightarrow \bar{z} = -in\pi$

$\sin(i\bar{z}) = \sin i(-in\pi) = \sin(-i^2 n\pi)$

$\sin(i\bar{z}) = \sin n\pi$

$\sin(i\bar{z}) = 0$

$\overline{\sin(iz)} = \overline{\sin i(i n\pi)} = \overline{\sin(i^2 n\pi)}$

$= \overline{\sin(-n\pi)}$

$= \bar{0} = 0$

$\therefore \sin(i\bar{z}) = \overline{\sin(iz)}$ if $z = n\pi i$

$(q \rightarrow p)$

Periodicity of Hyperbolic function:-

(i) $\sinh z$ and $\cosh z$ is $2\pi i$

(ii) $\tanh z$ is πi

P is a period of any function '$F(x)$'

if $f(x+P) = f(x)$

i) $\cosh z$:-

taking $\cosh(z+2\pi i) = \frac{e^{z+2\pi i} + e^{-(z+2\pi i)}}{2}$

$= \frac{e^{z} \cdot e^{2\pi i} + e^{-z} e^{-2\pi i}}{2} =$

$\left\{ \begin{array}{l} e^{i 2\pi} = \cos 2\pi + i\sin(2\pi) \\ = 1 + i(0) = 1 \\ e^{-2\pi i} = 0 \end{array} \right.$

$= \frac{e^{z}\cdot(1) + e^{-z}(1)}{2} = \frac{e^{z}+e^{-z}}{2}$

$\cosh z = \frac{e^{z}+e^{-z}}{2}$

If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, then

Show that $\cos^2\theta = \pm \sin\alpha$

given that

$\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$

$\sin\theta \cos i\phi + \cos\theta\, i\sin\phi = \cos\alpha + i\sin\alpha$

$\sin\theta \cosh\theta + \cos\theta\, i\sinh\theta = \cos\alpha + i\sin\alpha$

$\sin\theta\cosh\theta + i\cos\theta\sinh\phi = \cos\alpha + i\sin\alpha$

By equating real and Imp:-

$\sin\theta\cosh\phi = \cos\alpha$; $\cos\theta\sinh\phi$

$\cosh\phi = \dfrac{\cos\alpha}{\sin\theta}$; $\sinh\phi = \dfrac{\sin\alpha}{\cos\theta}$

Squaring and Subtracting :-

$$\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin\alpha\sin^2\theta$$

$$(1-\cos^2\theta)\cos^2\theta = (1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha\,(1-\cos^2\theta)$$

$$\cancel{\cos^2\theta} - \cos^4\theta = \cancel{\cos^2\theta} - \cancel{\sin^2\alpha\cos\theta} - \sin^2\alpha + \cancel{\sin^2\alpha\cos^2\theta}$$

$$-\cos^4\theta = -\sin^2\alpha$$

$$\cos^4\theta = \sin^2\alpha$$

$$\sqrt{\cos^4\theta} = \sqrt{\sin^2\alpha}$$

$$\cos^2\theta = \pm\sin a$$

Lecture # 10:-

Inverse Trigonometric and Hyperbolic Functions

1- one-to-one function:-

Inverse function:-

function:-

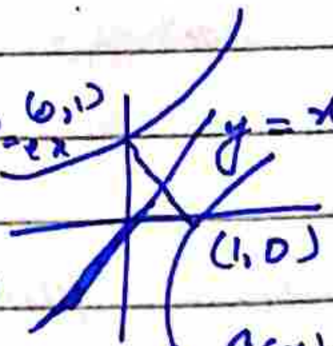
Relation b/w independent & dependent Variables.

$A, B \neq 0$, $R \subseteq A \times B = \{(x, y) \mid x \in A, y \in B\}$

If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
Then 'f' is one-one function.

2. Inverse function:-

If a function is 1-1, then it has an inverse



inverse of $e^x = \log x =$ reflection $f(x) = e^x$ of e^x about $y = x$

Inverse of $\log x = e^x =$ reflection of $\log_x$ about y-axis.

$y = f(x) \Rightarrow x = f^{-1}(y)$

Dom f = Rage$^{-1} f$

Rang f = Dom$^{-1} f$

3. Inverse Trigonometric Functions:-

$f(x) = \sin^{-1} x$ inverse Function

$y = \sin x$

$y = \sin(x + 2k\pi)$; $k \in \mathbb{Z}$

$\sin^{-1} y = x + 2k\pi \Rightarrow x = \sin^{-1} y - 2k\pi$

$x = \sin^{-1} y + 2(-k)\pi$

$\Rightarrow \boxed{x = \sin^{-1} y + 2n\pi}$

$n = -k \in \mathbb{Z} = \{0, \mp 1, \mp 2 \ldots\}$

4. Inverse Trigonometric and hyperbolic Functions for Complex variables:-

For $z \in \mathbb{C}$;

if $w = \sin z$; then we define $z = \sin^{-1}(x)$, i.e called inverse Sine function for complex variables.

$\cos^{-1} w$, $\sec^{-1} w$, $\tan^{-1} w$, $\cot^{-1} w$, $\operatorname{cosec}^{-1} w$; $w \in \mathbb{C}$

Hyperbolic functions :-

$\cosh^{-1} z$, $\sinh^{-1} z$, — $z \in \mathbb{C}$

$\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

5. Log ~~Inverse~~ expressions of Inverse Trigonometric functions:-

(i) $\sin^{-1} z = \frac{1}{i} \log\left(iz + \sqrt{1-z^2}\right)$

(ii) $\cos^{-1} z = \frac{1}{i} \log\left(z + \sqrt{z^2-1}\right)$

(iii) $\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$

(iv) $\cot^{-1} z = \frac{1}{2i} \log\left(\frac{z+i}{z-i}\right)$

(v) $\sec^{-1} z = \frac{1}{i} \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

(vi) $\operatorname{cosec}^{-1} z = \frac{1}{i} \log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

(v) $\sec^{-1} z = \frac{1}{i} \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $w = \sec^{-1} z$

$\sec w = z \quad ; z, w \in \mathbb{R}$

$\frac{1}{\cos w} = z \Rightarrow \cos w = \frac{1}{z}$

$\frac{e^{iw} - e^{-iw}}{2} = \frac{1}{z}$

$e^{iw} - e^{-iw} = \frac{2}{z} \Rightarrow e^{iw} - \frac{1}{e^{iw}} = \frac{2}{z}$

$$\frac{(e^{i\omega})^2 - 1}{e^{i\omega}} = \frac{2}{z}$$

$$z(e^{i\omega})^2 - z = 2e^{i\omega}$$

$$z(e^{i\omega})^2 - 2e^{i\omega} - z = 0$$

i.e quad in $e^{i\omega}$

$$e^{i\omega} = \frac{+2 \pm \sqrt{(-2)^2 - 4(z)(z)}}{2z}$$

$$= \frac{2 \pm \sqrt{4 - 4z^2}}{2z}$$

$$= \frac{2 \pm 2\sqrt{1 - z^2}}{2z}$$

$$= \frac{1 \pm \sqrt{1 - z^2}}{z}$$

$$e^{i\omega} = \frac{1 + \sqrt{1 - z^2}}{z}$$

$$i\omega = \log \frac{1 + \sqrt{1 - z^2}}{z}$$

$$\omega = \frac{1}{i} \log \frac{1 \pm \sqrt{1 - z^2}}{z}$$

6- Inverse Trigonometric Function (Application)

Seperate into real and Imaginary parts:-

(i) $Tan^{-1}(x+iy)$, (ii) $Cos^{-1}(\cos\theta + i\sin\theta)$

(iii) $Sin^{-1}(\cos\theta + i\sin\theta)$, (iv) $tan^{-1}(\cos\theta + i\sin\theta)$

(iii) $Sin^{-1}(\cos\theta + i\sin\theta)$

let $\sin^{-1}(\cos\theta + i\sin\theta) = (x+iy) \in \mathbb{R}$

$\cos\theta + i\sin\theta = \sin(x+iy)$

$\because \{ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos\theta + i\sin\theta = \sin x\cos(iy) + \cos x\sin(iy)$

$= \sin x\cosh y + \cos x \, i\sinh y$

Comparing real and Imaginary parts:-

$\cos\theta = \sin x\cosh y \to (1)$, $\sin\theta = \cos x\sinh y \to (ii)$

(1) $\Rightarrow \sin x = \dfrac{\cos\theta}{\cosh y}$, (2) $\Rightarrow \cos x = \dfrac{\sin\theta}{\sinh y}$

$\{ \sin^2 x + \cos^2 x = 1 \to (A)$

Putting $\sin x$ & $\cos x$ in (A)

(A) $\Rightarrow \dfrac{\cos^2\theta}{\cosh^2 y} + \dfrac{\sin^2\theta}{\sinh^2 y} = 1$

$$\Rightarrow \frac{\cos^2\theta \sinh^2 y + \sin^2\theta \cosh^2 y}{\cosh^2 y \cdot \sinh^2 y} = 1.$$

$$\cos^2\theta \sinh^2 y + \sin^2\theta \cosh^2 y = \cosh^2 y \sinh^2 y$$

$$(1-\sin^2\theta)\sinh^2 y + \sin^2\theta(1+\sinh^2 y) = (1+\sinh^2 y)(\sinh^2 y)$$

$$\sinh^2 y - \cancel{\sin^2\theta \sinh^2 y} + \sin^2\theta + \cancel{\sin^2\theta \cdot \sinh^2 y} = \cancel{\sinh^2 y} + \sinh^4 y$$

$$\sin^2\theta = \sinh^4 y$$

$$\sqrt{\sin^2\theta} = \sqrt{\sinh^4 y}$$

$$\sinh^2 y = \cancel{\pm}\sin\theta$$

$$\sinh y = \sqrt{\sin\theta}$$

$$\sinh y = \sqrt{\sin\theta}$$

$y = \sinh^{-1}\sqrt{\sin\theta}$ ✓ It is also true

$$\therefore \boxed{y = \log\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}}$$

↓ Imaginary part.

Now ① $\Rightarrow \cosh y = \dfrac{\cos\theta}{\sin x}$; $\sinh y = \dfrac{\sin\theta}{\cos x}$

$\{ \cosh^2 y - \sinh^2 y = 1 \rightarrow$ (A)

Putting in eq:-

$$\frac{\cos^2\theta}{\sin^2 x} - \frac{\sin^2\theta}{\cos^2 x} = 1$$

$$\Rightarrow \frac{\cos^2\theta\cos^2 x - \sin^2\theta\sin^2 x}{\cos^2 x\sin^2 x} = 1$$

$$\cos^2\theta\cos^2 x - \sin^2\theta\sin^2 x = \cos^2 x\sin^2 x$$

$$\cos^2 x\cos^2 x - (1-\cos^2\theta)(1-\cos^2 x) = \cos^2 x(1-\cos^2 x)$$

$$\cos^2\theta\cos^2 x - (1-\cos^2\theta - \cos^2\theta + \cos^2\theta\cdot\cos^2\theta)$$
$$= \cos^2\theta - \cos^4 x$$

$$\cancel{\cos^2\theta\cos^2 x} - 1 + \cancel{\cos^2 x} + \cos^2\theta - \cos^2\theta\cos^2 x$$
$$= \cancel{\cos^2 x} - \cos^4 x$$

$$-1 + \cos^2\theta = -\cos^4 x$$

$$\cancel{-}1(1-\cos^2\theta) = \cancel{-}\cos^4 x$$

$$\sin^2\theta = \cos^4 x$$

$$\sqrt{\sin^2\theta} = \sqrt{\cos^4 x}$$

$$\sin\theta = \cos^2\theta$$

$$\cos^2 x = \sin\theta$$

$$\sqrt{\cos^2 x} = \sqrt{\sin\theta}$$

$$\cos x = \pm\sqrt{\sin\theta} \quad \text{Real part.}$$

8. Log expression for inverse Hyperbolic Functions:-

Show that :-

(i) $\sinh^{-1}z = \log\left(z + \sqrt{z^2+1}\right)$

(ii) $\cosh^{-1}z = \log\left(z + \sqrt{z^2-1}\right)$

(iii) $\tanh^{-1}z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

(iv) $\coth^{-1}z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$

(v) $\operatorname{sech}^{-1}z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

(vi) $\operatorname{cosech}^{-1}z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right)$

(iii) $\tanh^{-1}z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

let :- $\tanh^{-1}z = w$

$z = \tanh w$

$\left\{ \tanh(w) = \frac{e^{(w)} - e^{-(w)}}{e^{(w)} + e^{-(w)}} \right.$

$\therefore\ z = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$

$\left\{ \text{if } \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d} \right.$
componendo & dividendo

$$\frac{z+1}{z-1} = \frac{(e^{w}-e^{-w})+(e^{w}+e^{-w})}{(e^{w}-e^{-w})-(e^{w}+e^{-w}}$$

$$\frac{z+1}{z-1} = \frac{\cancel{2}e^{w}}{-\cancel{2}e^{-w}}$$

$$\frac{z+1}{z-1} = \frac{e^{w}}{e^{-w}}$$

$$\frac{z+1}{z-1} = -e^{w+w} \Rightarrow \frac{z+1}{z-1} = -e^{2w}$$

$$(-1)\frac{z+1}{z-1} = e^{2w}$$

$$\frac{z+1}{1-z} = e^{2w}$$

$$e^{2w} = \frac{1+z}{1-z}$$

$$2w = \log\left(\frac{1+z}{1-z}\right)$$

$$w = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

$$\boxed{\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)}$$ proved

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Lecture # 11.

Complex Logarithmic Functions

1- Logarithmic Function in R:-

$2^3 = 8$

$\log_2(8) = 3$

$(10)^4 = 10\,000$

$\log_{10}(10\,000) = 4$

if

$$\boxed{y = a^x \iff \log_a y = x}$$

2- Special Cases of log Function in R.

$e^x = y \Rightarrow \log_e(y) = x$

$\ln y = x$

3. logarithmic function in Complex plane:-

if $z, w \in \mathbb{C}$; Such that $e^w = z$; Then we define natural log of $z \in \mathbb{C}$;

$\log z = w \rightarrow (1)$

$\left\{ \begin{array}{l} e^w = z \\ \Rightarrow \log z = w \end{array} \right.$

let $z = x + iy$, $w = u + iv$

$(1) \Rightarrow$

$\log(x+iy) = u + iv \rightarrow (2)$

$x+iy = e^{u+iv} \Rightarrow x+iy = e^u \cdot e^{iv}$

$x+iy = e^u [\cos v + i\sin v]$

$x+iy = e^u [\cos(v+2k\pi) + i\sin(v+2k\pi)] \rightarrow (2)$

Taking Modulus:-

$|x+iy| = |e^u| \cdot |\cos(x+2k\pi) + i\sin(v+2k\pi)|$

$\sqrt{x^2+y^2} = e^u \cdot \sqrt{\cos^2(v+2k\pi) + \sin^2(v+2k\pi)}$

$\sqrt{x^2+y^2} = e^u = z$

$u = \ln|z|$ Real valued function

$(2) \Rightarrow x+iy = e^u \cos(v+2k\pi) + i e^u \sin(v+2k\pi)$

$Re\, z = e^u \cos(v+2k\pi) \rightarrow (c)$

$Im\, z = y = e^u \sin(v+2k\pi) \rightarrow (d)$

by $(d) \div (c)$

$$\frac{e^x \sin(v+2k\pi)}{e^u \cos(v+2k\pi)} = \frac{y}{z}$$

$\Rightarrow \tan(v+2k\pi) = y/x$

$v + 2k\pi = \tan^{-1}\left(\frac{y}{x}\right)$

$v = \tan^{-1}\left(\frac{y}{x}\right) - 2k\pi$

$v = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$

Now $(2) \Rightarrow$

$\log(x+iy) = \ln|z| + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right]$

Multivalued

function

if $n=0$; Then

$$\boxed{\log(x+iy) = \ln|z| + i\,\mathrm{Arg}\,z}$$

Principal logrithm of $z \in \mathbb{C}$

4. Complex Log Function example 01

Find $\log(1-i)$.

given $z = 1-i = 1+i(-1)$

$x=1, \quad y=-1$

$\because \{\log(x+iy) = \ln|z| + i\,\mathrm{Arg}(x+iy)\}$

Now $|x+iy| = |1+i(-1)|$

$= \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\mathrm{Arg}(1-i) = \tan^{-1}\left(\frac{-1}{1}\right)$

$= -\tan^{-1}\left(\frac{1}{1}\right)$ $\because$ T-ray of Arg z lies in '4'

$= -\tan^{-1}(1) = -45^\circ = 45 \times \frac{\pi}{180}$

$= -\pi/4$

$\therefore \log(1-i) = \ln(\sqrt{2}) + i(-\pi/4)$

$= \ln(2)^{1/2} - i\pi/4$

$\log(1-i) = \frac{1}{2}\log 2 - i\pi/4$

$|z| = \sqrt{(\sqrt{3})^2 + (1)^2}$

$= \sqrt{4}$

$= 2$

5. Complex log Function (example-2)

$\log x = ?$; $x < 0$

$\because \log z = \log|z| + i \operatorname{Arg} z \rightarrow (1)$

Here $z = x = x + i(0)$; $x < 0$

$|z| = |x + i(0)|$

$|z| = |x| = -x$ if $x < 0$

$\operatorname{Arg} z = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{0}{-x}\right)$

$= \tan^{-1}(-0)$

$= \pi$

$\therefore (1) \Rightarrow \log z = \log(-x) + i\pi$

6. Complex log Function (example-3)

$\log(1 + \cos\theta + i\sin\theta) = \ln\left(2 \cdot \cos\left(\frac{\theta}{2}\right)\right) + i\left(\frac{\theta}{2}\right)$

$\log z = \log_{10}|z| + i \operatorname{Arg} z \rightarrow (1)$

given that :-

$z = (1 + \cos\theta) + i(\sin\theta)$

$|z| = \sqrt{(1 + \cos\theta)^2 + \sin^2\theta}$

$= \sqrt{1 + \cos^2\theta + 2\cos\theta + \sin^2\theta}$

$= \sqrt{1 + 1 + 2\cos\theta}$

$= \sqrt{2 + 2\cos\theta} = \sqrt{2(1 + \cos\theta)}$

$\left\{ 1 + \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) \right.$

$$|z| = \sqrt{2 \cdot 2\cos(\theta/2)} = 2\cos\theta/2$$

$$\operatorname{Arg} z = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tan^{-1}\left[\frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\cos^2\theta/2}\right]$$

$$= \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$= \tan^{-1}\tan(\theta/2) = \theta/2$$

Now (1) $\Rightarrow$

$$\boxed{\log z = \ln[(2\cos\theta/2)] + i\,[\theta/2]}$$

Lecture # 12:-

Complex Powers

1- Exponential and log function in $\mathbb{R}$:-

* e (Euler's constant)

$\forall\ x \in \mathbb{R}$; $e^x \to$ continuous growth at 100% for 'x' unit of time

$\Rightarrow 1 \to$ Scaled version of any real no

$\Rightarrow e \to$ Scaled version of any growth ratio rate.

$\Rightarrow$ log $\Rightarrow$ about time

2- Imaginary Growth:-

rotational $\leftarrow$ $e^{i\pi} = -1$

100% growth

3- Complex power (Introduction):-

$$e = 1 \cdot e = \lim_{n \to \infty}\left(1 + \frac{100\%}{n}\right)^n = 2.713 \ldots = e$$

for any complex number : we define

$$z^w = e^{w \log z} = e^{w(\ln|z| + i\,\mathrm{Arg}\,z)}$$

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4- Complex powers (Application - 1)

Evaluate:-

(i) $i^i = e^{i\log i} = e^{i\log(0+1(i))}$

$\log(x+iy) = (\ln|z| + i\,Arg z)$

$= e^{i(\ln\sqrt{(0)^2+(1)^2}} + i\,Tan^{-1}(\frac{1}{0})$

$= e^{i\ln 1 + i\tan^{-1}(\infty)}$

$\Rightarrow \begin{cases} \ln 1 = 0 \\ Tan^{-1}(\infty) = \pi/2 \end{cases}$

$= e^{i[(0)+i\pi/2]} = e^{i^2(\pi/2)} = e^{-\pi/2}$

5. Complex powers (Application-II)

(ii) a^i; $a>0$

$= e^{i\log a} = e^{i\log(a+i(0))}$

$e^{i\log a} = \cos(\log a) + i\sin(\log a)$

$= \cos(\ln a) + i\sin(\ln a)$.

6- Complex powers (Application III)

If $a^{\alpha+i\beta} = (x+iy)^{p+iq}$, $a>0$, prove that,-

(i) $\alpha = \frac{1}{2}p\log_a(x^2+y^2) - q\tan^{-1}\left(\frac{y}{x}\right)\log_a e$

(iii) $\log_a(x^2+y^2) = \alpha^4\left(\frac{\alpha p+\beta q}{p^2+q^2}\right)$

Solve: (i)

$$a^{\alpha+i\beta} = (x+iy)^{p+iq} \qquad \because z^w = e^{w\log z}$$

$$e^{(\alpha+i\beta)\log a} = e^{(p+iq)\log(x+iy)}$$

$$(\alpha+i\beta)\log a = (p+iq)\log(x+iy) + 2k\pi i \; ; k\in Z$$

for desired result ; put $k=0$

$$(\alpha+i\beta)\log a = (p+iq)\log(x+iy) \quad — (1)$$

$$\log a = \log a + i0 = \ln|a+i0| + i\tan^{-1}\left(\frac{0}{1}\right)$$
$$= \ln\sqrt{a^2+0^2} + i\tan^{-1}(0)$$
$$= \ln a + i(0)$$
$$= \ln a$$

$$\log(x+iy) = \ln|x+iy| + i\tan^{-1}\left(\frac{y}{x}\right)$$
$$= \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$
$$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

Now (1) $\Rightarrow$

$$(\alpha+i\beta)\ln a = (p+iq)\left[\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\alpha\ln a + i\beta\ln a = \frac{p}{2}\ln(x^2+y^2) + ip\tan^{-1}\left(\frac{y}{x}\right) + i\frac{q}{2}\ln(x^2+y^2) - q\tan^{-1}\left(\frac{y}{x}\right)$$

Comparing real & Imaginary parts:-

$$\alpha \ln a = \frac{P}{2}\ln(x^2+y^2) - q\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) \rightarrow A$$

$$\beta \ln a = P\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) + q_{1/2}\ln(x^2+y^2) \rightarrow B$$

from eq A

$$\alpha = \frac{P}{2}\frac{\ln(x^2+y^2)}{\ln a} - q\,\mathrm{Tan}^{-1}\frac{y}{x}\cdot\frac{1}{\ln a}$$

$$\left\{\frac{\ln A}{\ln\beta} = \frac{\log_e A}{\log_e \beta} = \log_\beta A \;\middle|\; \frac{1}{\log_a} = \frac{\log_e e}{\log_e a} = \log_a e\right.$$

$$\alpha = \frac{P}{q}\log_a(x^2+y^2) - q\,\mathrm{Tan}^{-1}\frac{y}{x}\cdot\log_a e$$

eq B

$$\beta = P\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)\cdot\frac{1}{\ln a} + \frac{q}{2}\frac{\ln(x^2+y^2)}{\ln a}$$

$$\beta = P\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)\log_a e + \frac{q}{2}\log_a(x^2+y^2)$$

Now taking:-

$$P\alpha + q\beta = P\left[\frac{P}{2}\log_a(x^2+y^2) - q\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)\log_a e\right] + q\left[P\,\mathrm{Tan}^{-1}\frac{y}{x}\log_a(x^2+y^2)\right]$$

$$= \frac{P^2}{2}\log_a(x^2+y^2) - Pq\,\cancel{\mathrm{Tan}^{-1}}\left(\frac{y}{x}\right)\cdot\log_a e + P\cancel{q\,\mathrm{Tan}^{-1}}\left(\frac{y}{x}\right)\log_a e\,(x^2+y^2)$$

$= \log_a(x^2+y^2)\left[\frac{p^2}{2} + \frac{q^2}{2}\right]$

$= \log_a(x^2+y^2)\left(\frac{p^2+q^2}{2}\right)$

$p\alpha + q\beta = \log_a(x^2+y^2)\left(\frac{p^2+q^2}{2}\right)$

$$\frac{2(p\alpha+q\beta)}{p^2+q^2} = \log_a(x^2+y^2)$$

7 Square root Issue:-

What is $\sqrt{4}$? ± 2 or '2' -- 1!!

if $\sqrt{4} = 2$

$-\sqrt{4} = -2$

$\pm\sqrt{4} = \pm 2$ $\qquad \sqrt{4} \neq \pm 2 \ \forall x \in \mathbb{R}$

for complex number:-

$\sqrt{4} = \sqrt{4 \cdot 1}$

$= \sqrt{4(\cos 0 + i\sin 0)}$

$= \sqrt{4(\cos(0+2k\pi) + i\sin(0+2k\pi))}$

$= \left[4(\text{cis}(2k\pi))\right]^{1/2}$

$= (4)^{1/2}\left[\text{cis}(2k\pi)\right]^{1/2} = 2$

$= (4)^{1/2}\,\text{cis of } k\pi$

$k=0$

$\sqrt{4} = 2 cis(0\pi)$

$\sqrt{4} = 2(\cos 0 + i \sin 0)$

$\sqrt{4} = \pm 2$.

for complex number

$\sqrt{4} = \pm 2$

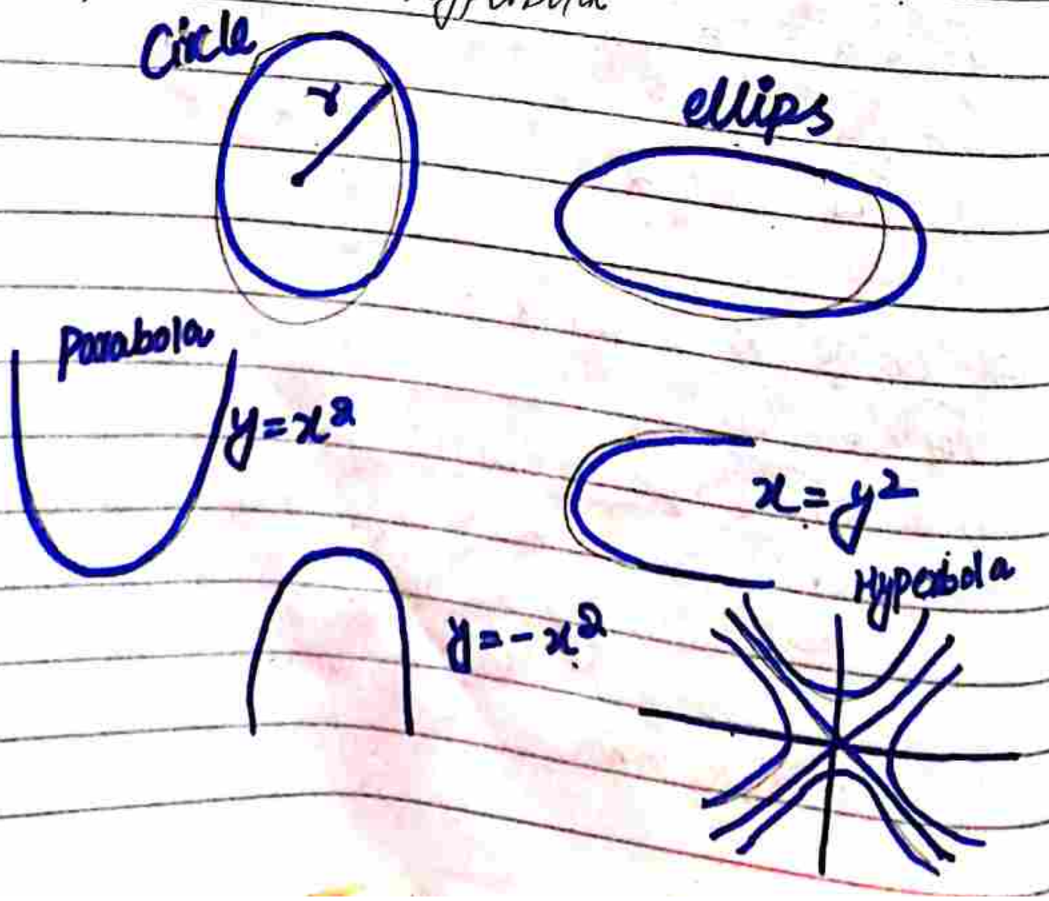
Lecture # 13

Introduction to Conic Section

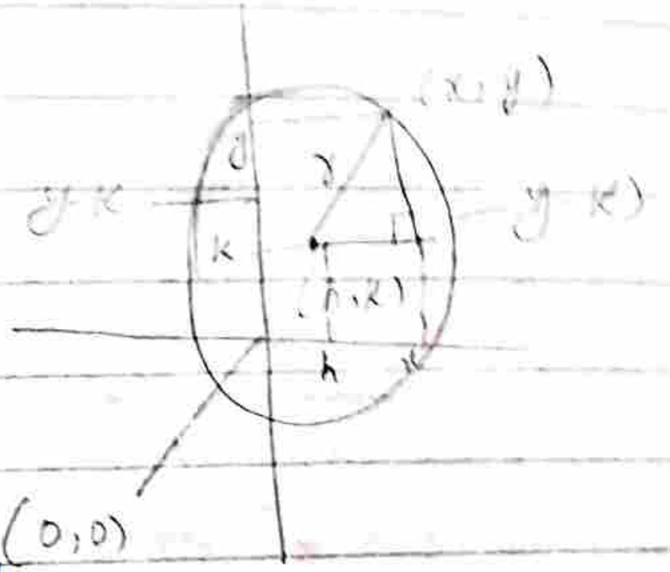
Circle and its General Equation

Conic function

- Circle
- ellips
- Parabola
- hyperbola



Circle (Introduction)



general equation of circle:-

$$\boxed{r^2 = (x-h)^2 + (y-k)^2}$$

Area of circle = area = πr^2

Circumference of circle = $2\pi r$

Lecture #14

Parabola and its Porperties

1- Introduction to Parabola:-

Parabola is defined as the set of all points that are equally distance from point and line.

2- Example of Parabola when the vertex is at origin:-

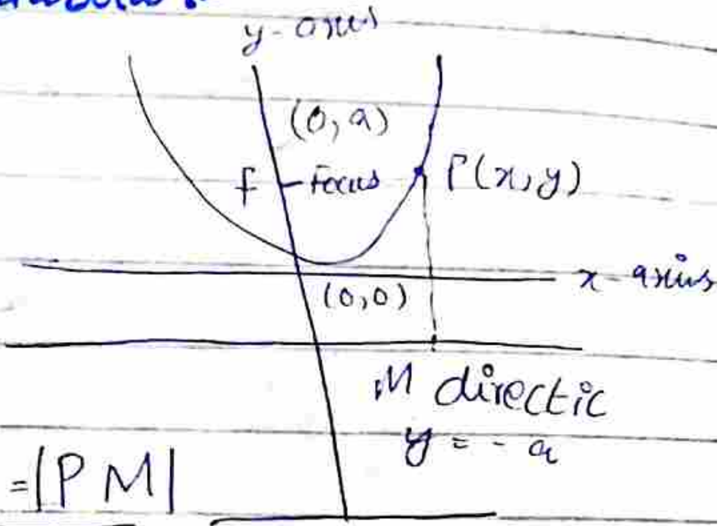
$y^2 = 8x$, $y^2 = 4ax$, $4a = 8$

$a = 2$

A line that passes through the focus and is perpendicular to the major ~~line~~ axis and has both point on the curve is called latusretum.

Equations of Parabola :-

vertex is at the origin.



$$|FP| = |PM|$$

$$\sqrt{(x-0)^2 + (y-a)^2} = \sqrt{((y-(-a))^2}$$

$$= (x-0)^2 + (y-a)^2 = (y+a)^2$$

$$\Rightarrow x^2 + y^2 + a^2 + 2ay = y^2 + a^2 + 2ay$$

$$x^2 = 2ay + 2ay$$

$\boxed{x^2 = 4ay}$ parabola equation when opening upward.

$\boxed{x^2 = -4ay}$ Parabola equation when opening doward.

* When parabola opeing right side on x-axis.

$$|FP| = |MP|$$

$$\sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-(-a))^2}$$

$$(x-a)^2 + y^2 = (x+a)^2$$

$$\cancel{x^2} + \cancel{a^2} - 2xa + y^2 = \cancel{x^2} + \cancel{a^2} + 2xa$$

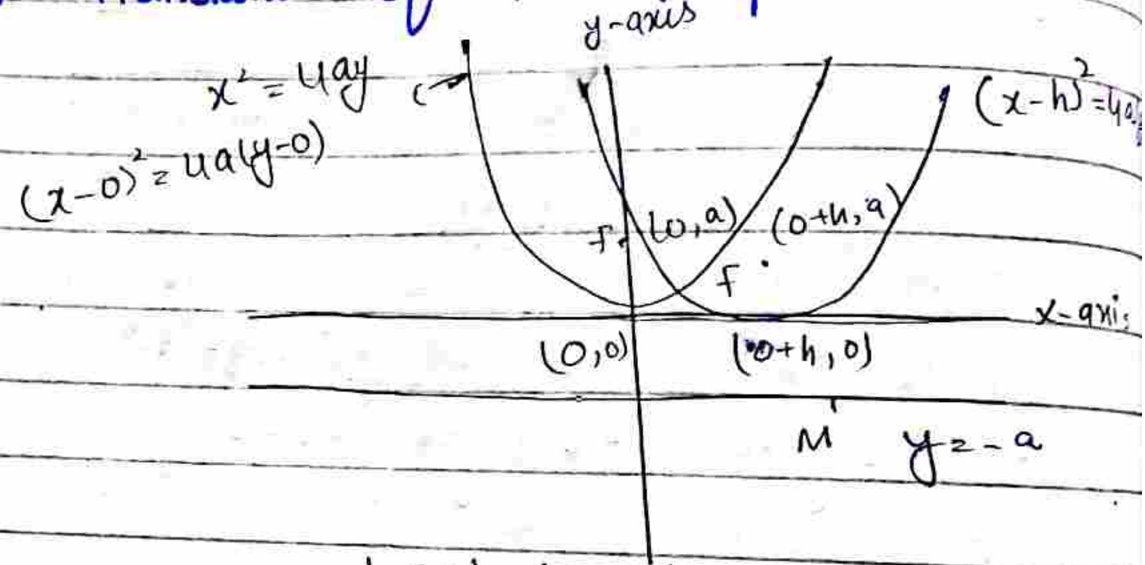
$$y^2 = 2xa + 2xa = \boxed{y^2 = 4ax}$$

when vertix at origin

when Parabola opening left Side.

$$y^2 = -4ax$$

4- Translation of axes in parabolas:-



$$|FP| = |PM|$$

$$\sqrt{(x-h)^2 + (y-a)^2} = \sqrt{(y-(-a))^2}$$

$$(x-h)^2 + (y-a)^2 = (y+a)^2$$

$$(x-h)^2 + \cancel{y^2} + \cancel{a^2} - 2ay = \cancel{y^2} + \cancel{a^2} + 2ay$$

$$(x-h)^2 = 2ay + 2ay = 4ay$$

$$\boxed{(x-h)^2 = 4ay}$$

ii) Parabola translate at 'k' unit.

Translation of axis in parabola:-

$(y-k)^2 = 4a(x-h)$

5. Example '1' of Translation of Axes:-

$2(x-3)^2 = 10(y+1) \rightarrow (i)$

Equation :-

$(x-h)^2 = 4a(y-k) \quad \rightarrow (ii)$

Divide by '2' eq (i)

$$\frac{\cancel{2}(x-3)^2}{\cancel{2}} = \frac{\overset{5}{\cancel{10}}(y+1)}{\cancel{2}}$$

$(x-3)^2 = (y+1) \rightarrow (iii)$

By comparing (ii) and (iii)

$(x-3)^3 = 5(y-(-1))$

vertex $(h,k) = (3,-1)$

focus $(3, -1+1.25) = (3, .25)$

$a(.2,5) = y = -2.25 \Rightarrow$ directix

$= \ln|z| + i\arg(z)$

Therefore $\log(1+\sqrt{3}i)$
$= \ln 2 + \frac{\pi}{3} i$ Ans.

Lecture # 15

Introduction to Ellipse

Radius is different in Ellipse in x and y direction.

* radius in x direction is Semi major axis.
* radius in y direction is Semiminhor axis.

Major equation :-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Lecture # 16:-

Hyperbola and its properties

1- Introduction to Hyperbola:-

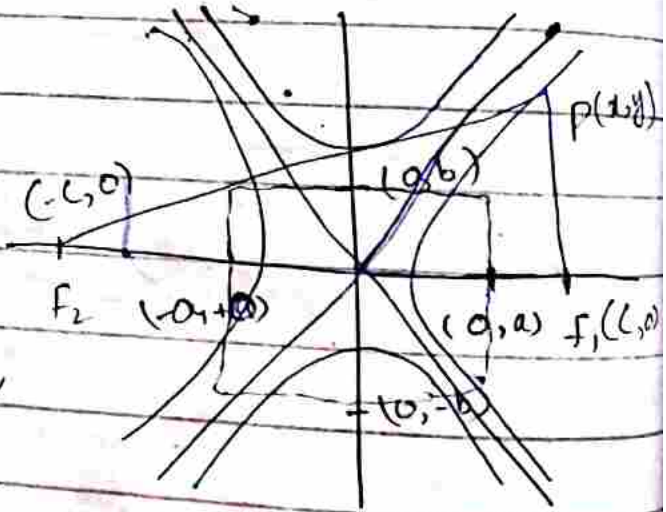
General equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

OR

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$|F_2P| - F_1P| = 2a$



When Hyperbola is facing up and down side then the equation is:-

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

When Hyperbola is facing left and right side then the equation is:-

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

2. Example of Hyperbola :-

. Sketch and Identify the focus :-

$\frac{x^2}{2^2} - \frac{y^2}{2^2}$ by comparing with

Starandard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

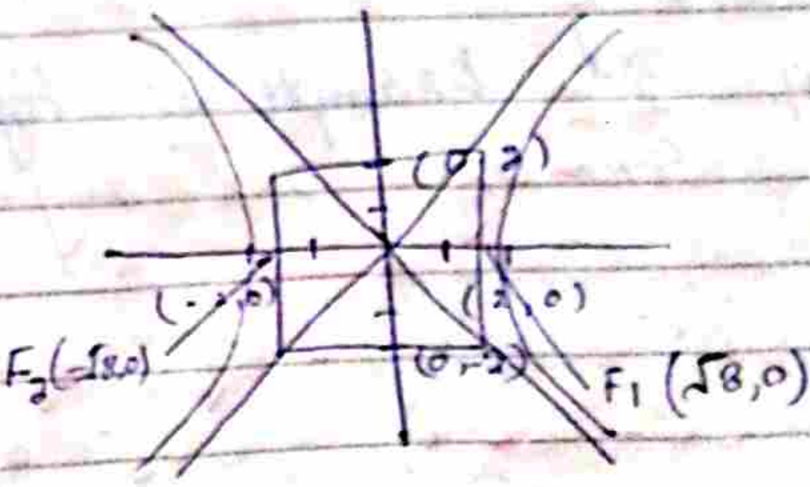
$a = 2, \quad b = 2$

$V_1(2,0)$, $V_2(-2,0)$

$c^2 = a^2 + b^2 = 4+4 = 8$

$c = \sqrt{8}$

$f_1(\sqrt{8},0)$, $f_2(-\sqrt{8},0)$



3. 2nd Example of Hyperbola:-

$9y^2 - 4x^2 = 144$, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$\frac{9y^2}{\cancel{144}\,16} - \frac{4x^2}{\cancel{144}\,36} = 1$

$\frac{y^2}{16} - \frac{x^2}{36} = 1$, $\frac{y^2}{4^2} - \frac{x^2}{6^2} = 1$

By comparing

$V_1(6,0)$, $V_2(-6,0)$

$V_3(0,4)$, $V_4(0,-4)$

$C^2 = a^2 + b^2 = 16 + 36 = 52$

$C = \sqrt{52} \simeq 7.2$

4. 3rd Example of Hyperbola:-

$9x^2 - 4y^2 - 36x - 24y - 36 = 0$

$9x^2 - 36x \quad - 4y^2 - 24y = 36$

$9(x^2 - 4x + 4) \quad -4(y^2 - 6y + 9) = 36 + 36 - 36$

$9(x-2)^2 - 4(y+3)^2 = 36$

$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

$\Downarrow$

Comparing

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

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$x' = x - 2, \quad y' = y + 3$

$\frac{(x')^2}{4} - \frac{y^2}{9} = 1$

$\frac{(x')^2}{2^2} - \frac{y'^2}{3^2} = 1$

$a = 2, \quad b = 3$

$c = \sqrt{a^2 + b^2} = \sqrt{4+9} = \sqrt{13}$

$f_1(\sqrt{13}, 0), \; f_2(-\sqrt{13}, 0)$

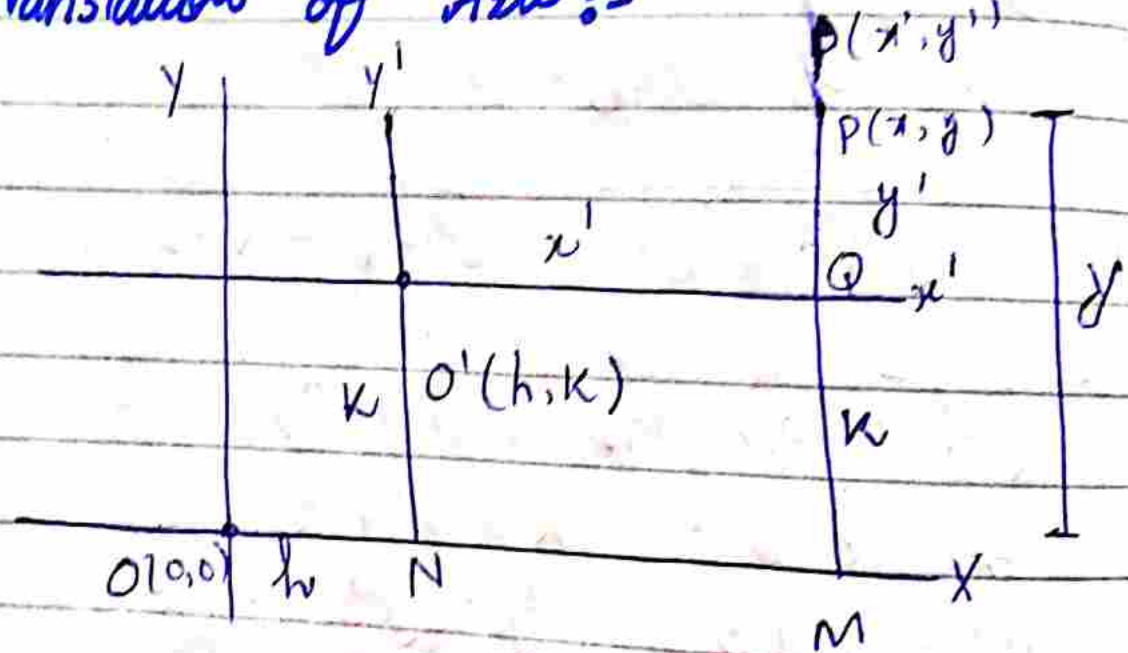
$x' = x - 2, \quad x = x' + 2$

$y' = y + 3, \quad y = y' - 3$

$f_1(\sqrt{13} + 2, -3), \; f_2 \sqrt{13} - 2, 3)$

Lecture # 17 :-
Translation and Rotation of Axis.

* Translation of Axis :-



$OM = x$

$ON = h$

$MN = x'$

$OM = ON + MN$

$\boxed{x = h + x'}$

$PM = y$

$PQ = y'$

$QM = K$

$PM = PQ + QM$

$\boxed{y = y' + K}$

Example of the transformation of Parabola:-

Example:-
Determine the focus, vertex and directix of the Parabola?

$x^2+6x-8y+17=0$

$x^2+6x=8y-17$

$x^2+2(x)(3)+3^2=8y-17+3^2$

$(x+3)^2=8y-8$

$(x+3)^2=8(y-1)$

$x'=x+3$

$y'=y-1$, $a=2$

$(x')^2=4(2)(y')$

$x^2=4ay$

Directix $y'=-a$

$y-1=-2$

$\boxed{y=-1}$

focus:- $(0, a)$

$x'=0$, $y'=a$

$x+3=0$, $y-1=2$

$x=-3$ $y=3$

$\boxed{f(-3,3)}$

Vertex $(0,0)$

$x'=0$ $y'=0$

$x+3=0$ $y-1=0$

$x=-3$ $y=1$

$\boxed{V(-3,1)}$

Transformation of ellipse:-

Show that $x^2+4y^2-4x+8y+4=0$ represent an ellipse. sketch its graph and find its foci and vertices.

$(x^2-4x)+(4y^2+8y)=-4$

$[x^2-2(x)(2)+2^2]+[(2y)^2+2(2y)(2)+2^2]$
$=-4+2^2+2^2$

$(x-2)^2+(2y+2)^2=24$

$(x-2)^2+2^2(y+1)^2=4$

Dividing by '4'

$\frac{(x-2)^2}{4}+\frac{4(y+1)^2}{4}=1$

$\frac{(x-2)^2}{2^2}+\frac{(y+1)^2}{1^2}=1$

Here $a=2$, $b=1$

$\therefore b^2=a^2-c^2$

$c^2=a^2-b^2=2^2-1^2=3$

$c=\pm\sqrt{3}$

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$$\frac{x''}{2} + \frac{y''}{1} = 1$$

Center $(0, 0)$

$x' = 0$, $y' = 0$

$x - 2 = 0$, $y + 1 = 0$

$x = 2$, $y = -1$

Center $(2, -1)$

Foci $(\pm c, 0)$

$x' = c$, $y' = 0$

$x - 2 = \pm\sqrt{3}$, $y + 1 = 0$

$x = 2 \pm \sqrt{3}$, $y = -1$

$f_1 (2 - \sqrt{3}, -1)$

$f_2 (2 + \sqrt{2}, -1)$

Major axis $A(\pm a, 0)$

$x' = \pm a$ | $y' = 0$

$x - 2 = \pm a$ | $y + 1 = 0$

$x - 2 = \pm 2$ | $y + 1 = 0$

$x = 2 \pm 2$ | $y = -1$

$A_1 = (2 - 2, -1)$, $A_2 (2 + 2, -1)$

$= (0, -1)$

$(4, -1)$

Minor axes (co-vertices)

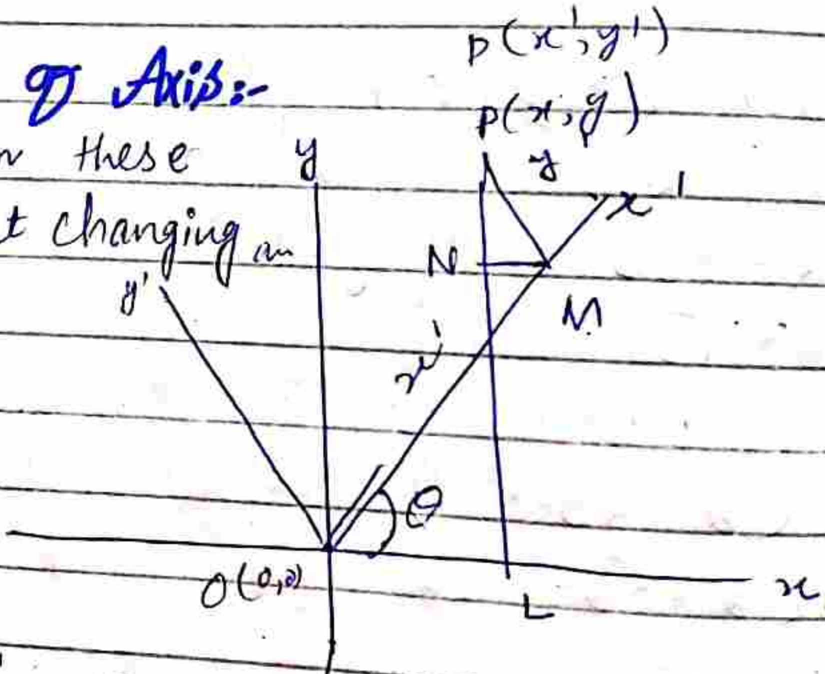
$B(0, \pm b)$

$x' = 0$	$y + 1 = \pm 1$
$x - 2 = 0$	$y = -1 + 1$
$x = 2$	$y = -1 - 1 = -2$, $y = -1 + 1 = 0$

$B_1(2, -2)$; $B_2(2, 0)$

• Rotation of Axis:-

we transform these axis without changing an origion.



$\angle XOX' = \theta = \angle NPM$

ΔNPM

$\sin\theta = \frac{NM}{PM}$

$NM = PM\sin\theta$

$NM = y'\sin\theta$

$\cos\theta = \frac{NP}{PM}$, $NP = PM\cos\theta$

$NP = y'\cos\theta$

Example:-

$9x^2 + 24y + 16y^2 - 125y = 0 \rightarrow (1)$

as we know that

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \rightarrow (2)$

Comparing (1) and (2)

$a = 9 \quad , \quad 2h = 24 \, , \, h = 12$

$b = 16 \quad , \quad 2g = 0 , \quad 2f = -125$

$c = 0$

$\Rightarrow h^2 - ab$

$= (12)^2 - (9)(16)$

$= 144 - 144 = 0$ its mean it is Parabola.

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

$\therefore \tan 2\theta = \frac{2h}{a-b} = \frac{2(12)}{9-16} = \frac{-24}{7}$

$\frac{2\tan\theta}{1-\tan^2\theta} = -24/7$

$\cancel{14\tan\theta = -24}$

$12\tan^2\theta - 7\tan\theta - 12 = 0$

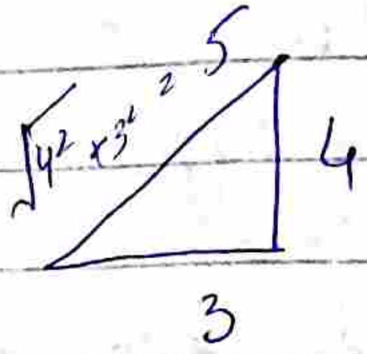
$\tan\theta = \frac{7 \pm \sqrt{(-7)^2 - 4(12)(-12)}}{2(12)}$

$\tan\theta = \frac{4}{3} , -\frac{3}{4}$

$\tan\theta = 4/3$

$\sin\theta = 4/5$

$\cos\theta = 3/5$



$$x = x'\left(\frac{3}{5}\right) - y'\left(\frac{4}{5}\right)$$

$$= \frac{3x' - 4y}{5} \rightarrow A$$

$$y = x'\left(\frac{4}{5}\right) + y'\left(\frac{3}{5}\right)$$

$$y = \frac{4x' + 3y'}{5} \rightarrow B$$

$$\left\{ x' = \frac{3x+4y}{5},\ y' = \frac{-4x+3y}{5} \right\}$$

Lecture #18:-

General Equation of Second Degree

A equation:-

$ax^2+by^2+2hxy+2gx+2fy+c=0$ —(1)

is Called homogenous equation of 2nd degree.

Where a, b, and h are not Simultaneously Zero.

$\because \left(\begin{matrix} 2gx+2fy+c=0 \\ a=0,\ b=0,\ h=0 \end{matrix} \right)$ Stright line eq

eq (1) will represent Parabola, ellipse and hyperbola:-

if

(i) $h^2-ab=0$ (Parabola)

(ii) $h^2-ab<0$ (ellipse)

(3) $h^2-ab>0$ (hyperbola)

Also that eq (1) will represent a Pair of stright line if its determents is equal to zero.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example:-

$$2x^2 - xy + 5x - 2y + 2 = 0$$

$a = 2,\ b = 0,\ c = 2,\ 2h = -1 \Rightarrow h = -\frac{1}{2}$

$2g = 5,\ g = \frac{5}{2},\ 2f = -2$

$f = -1$

$$\begin{vmatrix} 2 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 0 & -1 \\ \frac{5}{2} & -1 & 2 \end{vmatrix} = 0$$

$$2\left[0-1\right] + \frac{1}{2}\left[-1 + \frac{5}{2}\right] + \frac{5}{2}\left[\frac{1}{2}\right]$$

$$\Rightarrow -2 + \frac{3}{5} + \frac{5}{2} = 0$$

Example of 2nd degree equation

for what value of λ will each of the following equation represent a pair of stright line.

$$\lambda x^2 - 10xy - 12y^2 + 5x - 16y - 3 = 0 \quad \text{--- (1)}$$

As we know that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (2)}$$

Comparing eq (1) and (2)

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$a = \lambda$, $2h = -10$, $h = -5$

$b = 12$, $2g = 5$, $g = 5/2$

$2f = -16$, $f = -8$, $c = -3$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & -5 & 5/2 \\ -5 & 12 & -8 \\ 5/2 & -8 & -3 \end{vmatrix} = 0$$

$$\lambda(-36-64) + 5(15+20) + \frac{5}{2}(40-30) = 0$$

$$-100\lambda + 175 + 25 = 0$$

$$\frac{200}{100} = \lambda \Rightarrow \lambda = 2.$$

Angle from Stright line (Second degree)

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$\because$ if $a+b=0$; Then two st. lines are at right angle.

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Example:-

$$6x^2 + xy - y^2 - 21x - 8y + 9 = 0 \rightarrow (2)$$

$a = 6,\ b = -1,\quad 2h = 1,\ h = \frac{1}{2}$

$$\tan\theta = \frac{2\sqrt{(\frac{1}{2})^2 - 6(-1)}}{6(-1)} = \frac{2\sqrt{\frac{1}{4} + 6}}{5}$$

$$= \frac{2\sqrt{25/4}}{5} = \frac{2(5/2)}{5} = 1$$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4} = 45^\circ$$

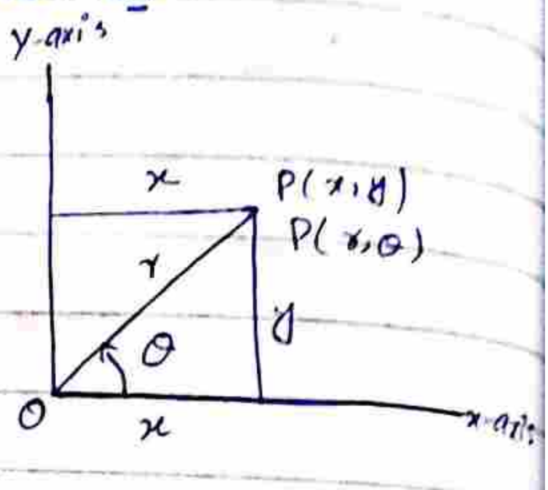
Lecture #19 MTH403

1- Relationship between Cartesian and Polar Coordinate Systems?

$\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$

$\boxed{x = r\cos\theta}$, $\boxed{y = r\sin\theta}$

$\boxed{r^2 = x^2 + y^2}$



$\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}$

$\tan\theta = \frac{y}{x}$

$\boxed{\theta = \tan^{-1}\frac{y}{x}}$

Example.-

Express the given equation in rectangular coordinates.

$$r = \frac{8}{2-\cos\theta}$$

$$r = \frac{8}{2-\cos\theta}$$

$2r - r\cos\theta = 8$

$2\sqrt{x^2+y^2} - x = 8$

$2\sqrt{x^2+y^2} = x + 8$

Sq on b.s

$4(x^2+y^2) = (x+8)^2$

$4x^2 + 4y^2 = x^2+16x+64$

$3x^2+4y^2-16x-64=0$

Example:-

Transform the eq $y = \frac{x}{x+1}$ in polar form:-

$y = \frac{x}{x+1}$

$xy + y = x$ $\quad \because x = r\cos\theta$

$y = r\sin\theta$

$r\cos\theta \cdot r\sin\theta + r\sin\theta = r\cos\theta$

$r^2 \sin\theta\cos\theta + r\sin\theta - r\cos\theta = 0$

Dividing by r $(r \neq 0)$

$r\sin\theta\cos\theta + \sin\theta - \cos\theta = 0$

Divinding b.s by $\sin\theta\cos\theta$

$r + \sec\theta - \csc\theta = 0$

Examples.

Find the Polar Corrdinates of the point P whose Rectangular points are $(3\sqrt{2}, -3\sqrt{2})$

Here $x = 3\sqrt{2}$, $y = -3\sqrt{2}$, $r, \theta = ?$

$r = \sqrt{x^2+y^2} = \sqrt{18+18} = \sqrt{36} = 6$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{3\sqrt{2}}{-3\sqrt{2}}\right)$

$\theta = \tan^{-1}(-1) = \frac{3\pi}{4}, \; 7\pi/4$

Hence the rectangular point in Polar cordinate System is $(r, \theta) = (6, \frac{7\pi}{4})$

Lecture # 20.

Polar equations of Conics.

Example:-

Transform the equation $y^2 = 4a(x+a)$ in Polar Corrdinates.

$y^2 = 4a(x+a) \rightarrow$ (I)

$x = r\cos\theta, \quad y = r\sin\theta$

$r^2 = x^2 + y^2$

Adding 'x^2' on b.s of eq (I)

$x^2 + y^2 = 4a(x+a) + x^2$

$r^2 = 4ax + 4a^2 + x^2$

$r^2 = (x+2a)^2$

$r^2 = (r\cos\theta + 2a)^2$

$r = \pm(r\cos\theta + 2a)$

$r = (r\cos\theta + 2a) \rightarrow$ (2), $r = -(r\cos\theta + 2a)$

$\Downarrow$

$r - r\cos\theta = 2a$

$r(1-\cos\theta) = 2a$

$$\boxed{r = \frac{2a}{1-\cos\theta}} \rightarrow (3)$$

Polar equation of the Conic.

Note:- $2a = l$ $\quad e = 1$

$$\boxed{r = \frac{l}{1 - e\cos\theta}}$$ → eq (4)

This equation ~~show~~ represents, parabola, hyperbola and ellipe.

⇒ eq (4) represent parabola when $e = 1$

eq (4) represent an ellipse when $e < 1$

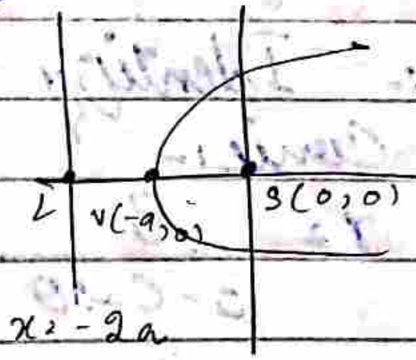
eq (4) represent an hyperbola when $e > 1$

where $l = 2a$ (length of semi l.r.) $\qquad |LM| = 4a$

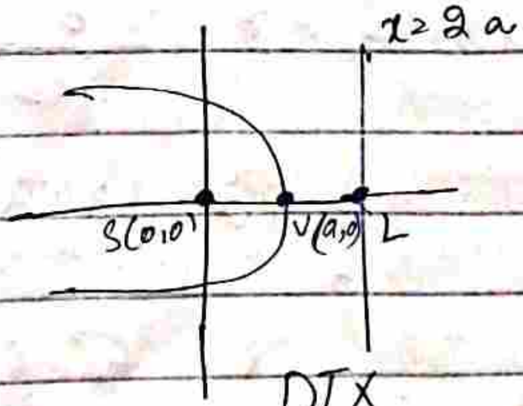
Some other Polar equation of Conic:-

$r = \dfrac{l}{1 - e\cos\theta}$ ⟹

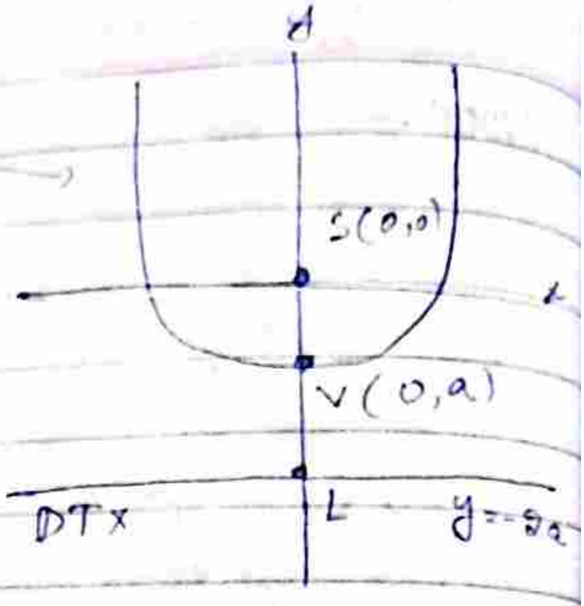
This can also be write as:-



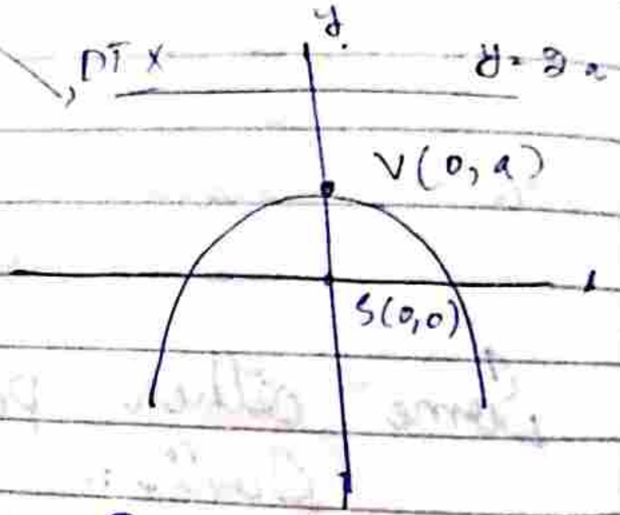
$r = \dfrac{l}{1 + e\cos\theta}$ ⟶



(iii) $r = \dfrac{l}{1 - e\sin\theta}$



(iv) $r = \dfrac{l}{1 + e\sin\theta}$



Example:- Identify & graph of the Conic:-

$$r = \frac{8}{3 - \cos\theta}$$

Sol:- $r = \dfrac{8}{3 - \cos\theta}$, $r = \dfrac{l}{1 - e\cos\theta}$

By Comparing

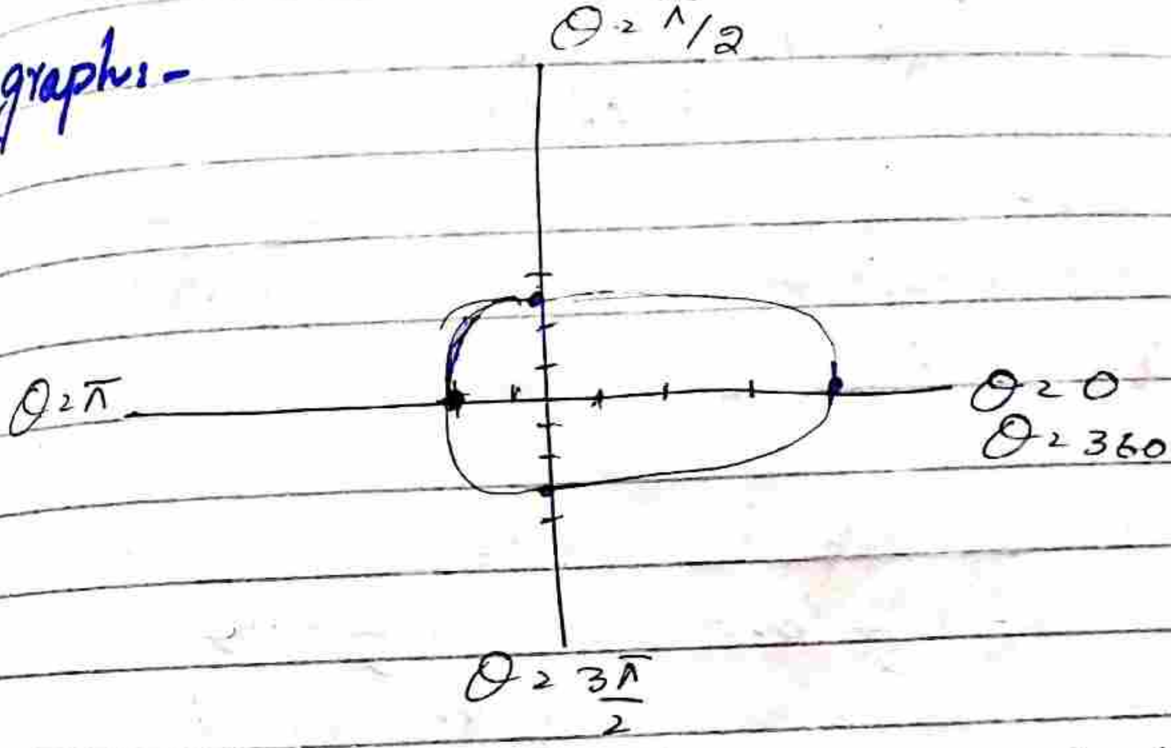
$$r = \frac{8}{3\left(1 - \frac{1}{3}\cos\theta\right)}$$

$$r = \frac{8/3}{1 - \frac{1}{3}\cos\theta}$$

Here $l = \frac{8}{3}$ = length of semi l.r.

$e = \frac{1}{3} < 1$ (Ellipse)

graph:-



$r = \frac{8/3}{3-\cos\theta} = \frac{8}{3-1} = 4 \because \cos\theta = 1$

$\theta = \frac{\pi}{2}, \quad r = \frac{8}{3-0} = 2.7$

$\theta = \frac{3\pi}{2}, \quad r = \frac{8}{3-0} = 2.7$

$\theta = \pi, \quad r = \frac{8}{3+1} = \frac{8}{4} = 2$

Example :-

Identify and graph the following equations.

$$r = \frac{10}{2+3\cos\theta}$$

$r = \dfrac{10}{2+3\cos\theta}$, $r = \dfrac{l}{1+e\cos\theta}$

$r = \dfrac{10}{2(1+\frac{3}{2}\cos\theta)}$

$r = \dfrac{10/2}{1+\frac{3}{2}\cos\theta}$ $\Rightarrow r = \dfrac{5}{1+\frac{3}{2}\cos\theta}$ comparing with $r = \dfrac{l}{1+e\cos\theta}$

$e = \frac{3}{2} > 1$ (Hyperbola)

$l = 5$ (length of semi l.r.)

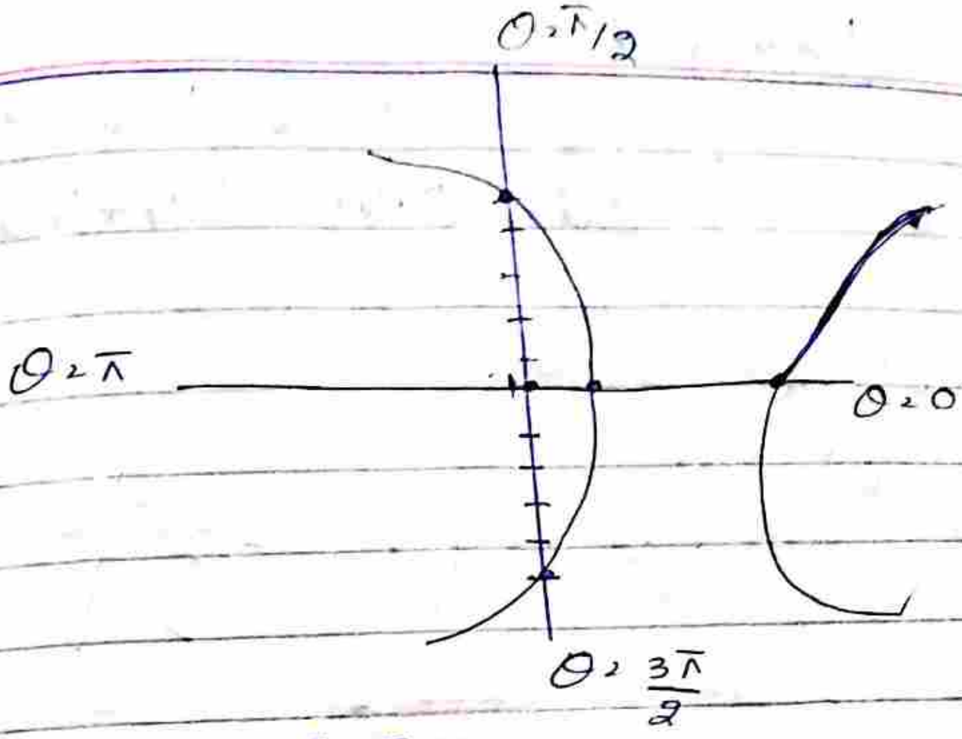
$r = \dfrac{10}{2+3\cos\theta}$

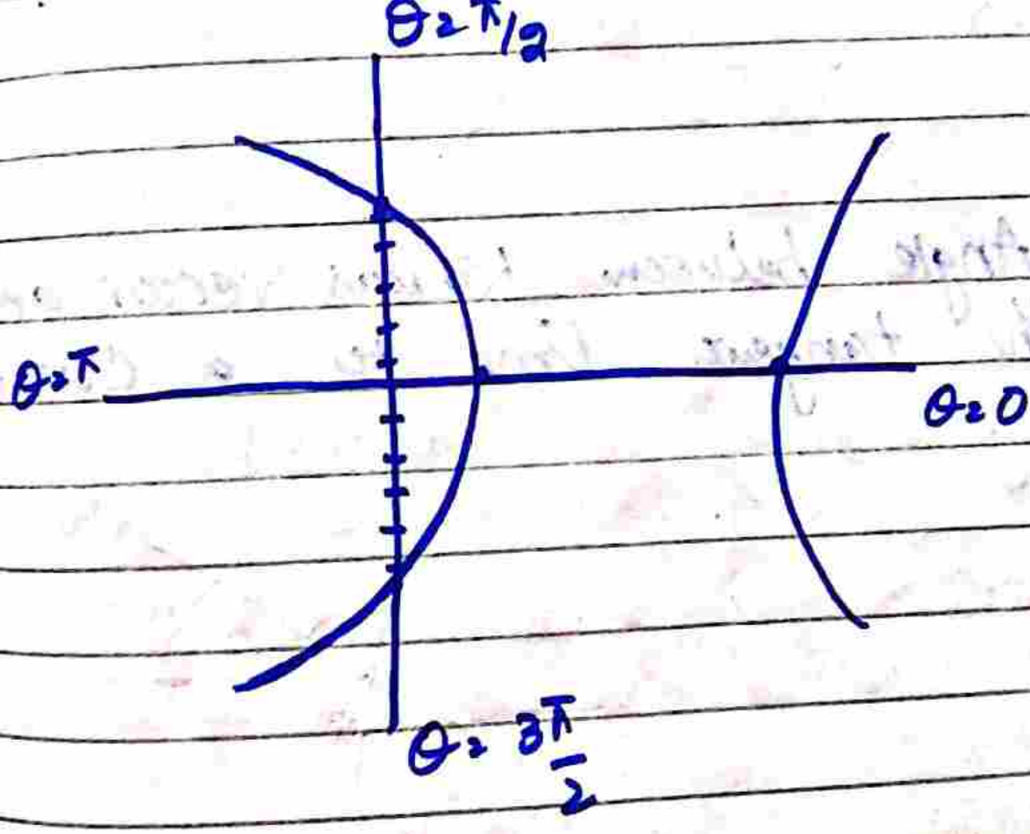
when $\theta = 0$, $r = \frac{10}{5} = 2$

$\theta = \frac{\pi}{2}$, $r = \frac{10}{2} = 5$

$\theta = \pi$, $r = \frac{10}{2-3} = -10$

$\theta = \frac{3\pi}{2}$, $r = \frac{10}{2} = 5$

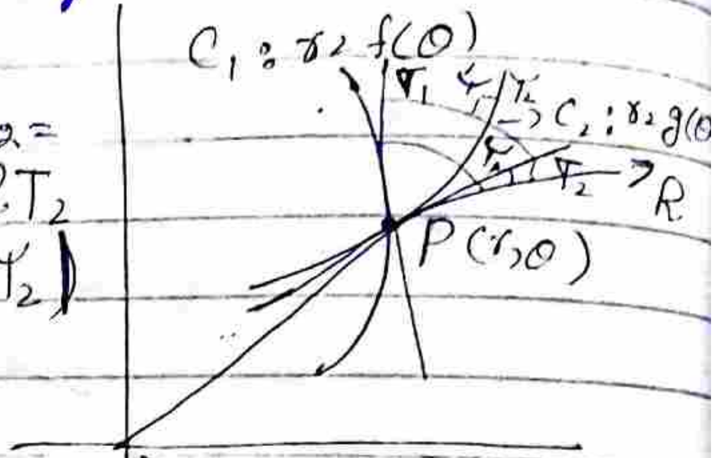

$\theta = \pi/2$
$\theta = \pi$
$\theta = 0$
$\theta = \frac{3\pi}{2}$


$\theta = \pi/2$
$\theta = \pi$
$\theta = 0$
$\theta = \frac{3\pi}{2}$

Lecture # 21

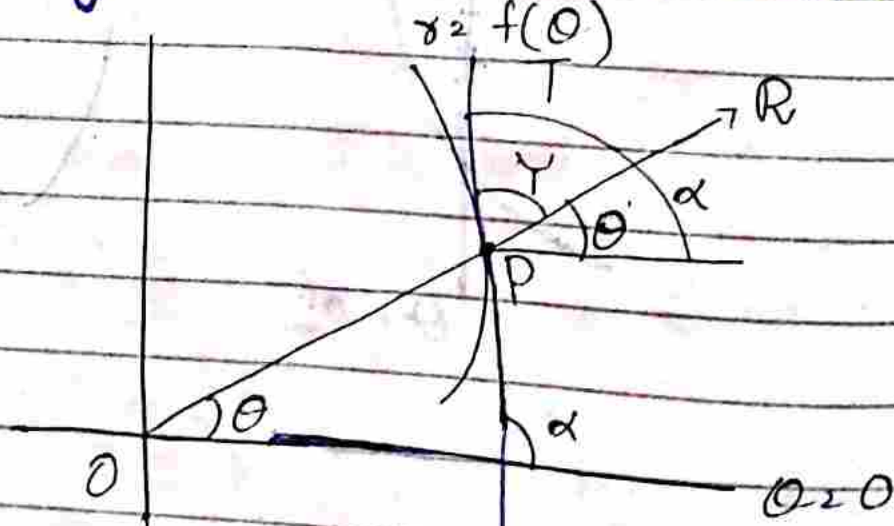
Angle between Two Curves in polar Coordinates.

Angle b/w C_1 and C_2 =
Angle b/w T_1 & T_2
$= |\psi_1 - \psi_2|$



ψ_1 is angle b/w T_1 & R.
ψ_2 is angle b/w T_2 is & R.

* Angle between Radius vector and the tangent line to a Curve



$\alpha = \theta + \psi$

Where:-

α is angle b/w T and $\theta = 0$
θ is angle b/w R and $\theta = 0$

ψ is angle between R and T

Theorem:-
prove that $\tan\psi = r\frac{d\theta}{dr}$

$\alpha = \theta + \psi$

$\psi = \alpha - \theta$

By applying 'tan' on both

$\tan\psi = \tan(\alpha - \theta)$

$= \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha \tan\theta} \rightarrow (1)$

α: angle b/w T and Inital line.

$\tan\alpha = m = \frac{dy}{dx}$, $\tan\theta = y/x$

By putting in eq (1)

$\tan\psi = \frac{dy/dx - y/x}{1 + \frac{dy}{dx} \cdot y/x} \rightarrow (2)$

$x = r\cos\theta$, $y = r\sin\theta$

$\left.\begin{aligned} \frac{dx}{dr} &= \cos\theta - r\sin\theta\frac{d\theta}{dr} \\ \frac{dy}{dr} &= \sin\theta + r\cos\theta\frac{d\theta}{dr} \end{aligned}\right]$ **By applying product rule here.**

By Chain rule:-

$$\frac{dy}{dx} = \frac{dy}{dr} \cdot \frac{dr}{dx}$$

$$\frac{dy}{dx} = \frac{\sin\theta + \cos\theta \, d\theta/dr}{\cos\theta - r\sin\theta \, d\theta/dr}$$

By putting values in eq (1)

$$\tan\psi = \frac{\sin\theta + \cos\theta \frac{d\theta}{dr}}{\cos\theta - r\sin\theta \frac{d\theta}{dr}} - \frac{\sin\theta}{\cos\theta}$$

$$\overline{1 + \frac{\sin\theta + r\cos\theta \, d\theta/dr}{\cos\theta - r\sin\theta \frac{d\theta}{dr}} \cdot \frac{\sin\theta}{\cos\theta}}$$

By Taking L.C.M.

$$\tan\psi = \frac{\cancel{\sin\theta\cos\theta} + r\cos^2\theta \, d\theta/dr - \cancel{\sin\theta\cos\theta} - r\sin^2\theta \, d\theta/dr}{\cos^2\theta - \cancel{r\sin\theta\cos\theta \frac{d\theta}{dr}} + \sin^2\theta + \cancel{r\sin\theta\cos\theta \frac{d\theta}{dr}}}$$

$$= \frac{r \, d\theta/dr \, (\cos^2\theta - \sin^2\theta)}{(\cos^2\theta - \sin^2\theta)}$$

$$= r \, d\theta/dr \, \frac{(1)}{(1)}$$

$$\tan\psi = r\frac{d\theta}{dr}$$ **Hence proved.**

Example:-
How to find angle between radius vector and tangent line.

* Find ψ of the following curves in polar coordinates.

(i) $r = a(1-\cos\theta)$.

(ii) $\frac{2a}{r} = 1+\sin\theta$

Solution:-

(i) $r = a(1-\cos\theta)$

$\frac{dr}{d\theta} = a\cdot(0-(-\sin\theta))$

$\frac{dr}{d\theta} = a\sin\theta$

Consider:- $\frac{r\,d\theta}{dr} = \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta}$

$\tan\psi = \frac{1-\cos\theta}{\sin\theta}$

$\tan\psi = \frac{\cancel{2}\sin^2\theta/2}{\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$ => Double angle formula
=> Half angle formula

$\tan\psi = \frac{\sin\theta/2}{\cos\theta/2}$

$\tan\psi = \tan\theta/2 \Rightarrow \boxed{\psi = \theta/2}$

(ii) $\frac{2a}{r} = 1 + \sin\theta$

$$r = \frac{2a}{1+\sin\theta}$$

$$r = 2a(1+\sin\theta)^{-1}$$

$$\frac{dr}{d\theta} = -2a(1+\sin\theta)^{-2}\cos\theta$$

$$\frac{dr}{d\theta} = -\frac{2a\cos\theta}{(1+\sin\theta)^2}$$

$$\tan\psi = r\frac{d\theta}{dr} = \frac{\cancel{2}a}{\cancel{1+\sin\theta}} \cdot \frac{(1+\sin\theta)^{\cancel{2}}}{-\cancel{2}a\cos\theta}$$

$$= -\frac{1+\sin\theta}{\cos\theta} = \cancel{\cos^2\theta/2} + \sin^2$$

$$\tan\psi = -\frac{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}{\cos^2\theta/2 - \sin^2\theta/2}$$

$$= -\frac{(\cos\theta/2 + \sin\theta/2)^{\cancel{2}}}{(\cos\theta/2 - \sin\theta/2)\cancel{(\cos\frac{\theta}{2} + \sin\theta/2)}}$$

$$= -\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}$$

=> Dividing by $\cos\frac{\theta}{2}$ in numeratier and denumarator.

$$\tan\psi = -\frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$\tan\psi = -\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$= \tan\left(\pi - \left(\frac{\pi}{4} + \theta/2\right)\right)$

$\tan\psi = \tan\left(\frac{3\pi}{4} - \frac{\theta}{2}\right)$

$$\boxed{\psi = 3\pi/4 - \theta/2}$$

Lecture # 22
Pedal Equation.

1 - Example How to find angle between two curves in polar Coordinate System - 2:-

*** Find the angle between the curves $r = a\theta$ and $r\theta = a$**

$r = a\theta \rightarrow (1)$, $\quad r\theta = a$

$r = a/\theta \rightarrow (2)$

By comparing (1) and (2)

$a\theta = a/\theta$

$a\theta^2 = a \Rightarrow \theta^2 = 1 \quad \because a \neq 0$

$\theta = \pm 1$

$r = a\theta$	$r = a/\theta$
$\frac{dr}{d\theta} = a$	$\frac{dr}{d\theta} = -a/\theta^2$
Consider	
$\frac{r\,d\theta}{dr} = a\theta \cdot \frac{1}{a} = \theta$	$\frac{r\,d\theta}{dr} = \frac{a}{\theta} \cdot -\frac{\theta^2}{a}$
$\tan\psi_1 = \theta$	$\tan\psi_2 = -\theta$

when $\theta = 1$, $\tan\psi_1 = 1$, $\tan\psi_2 = -1$

$\psi_1 = \tan^{-1}(1)$, $\psi_2 = \tan^{-1}(-1)$

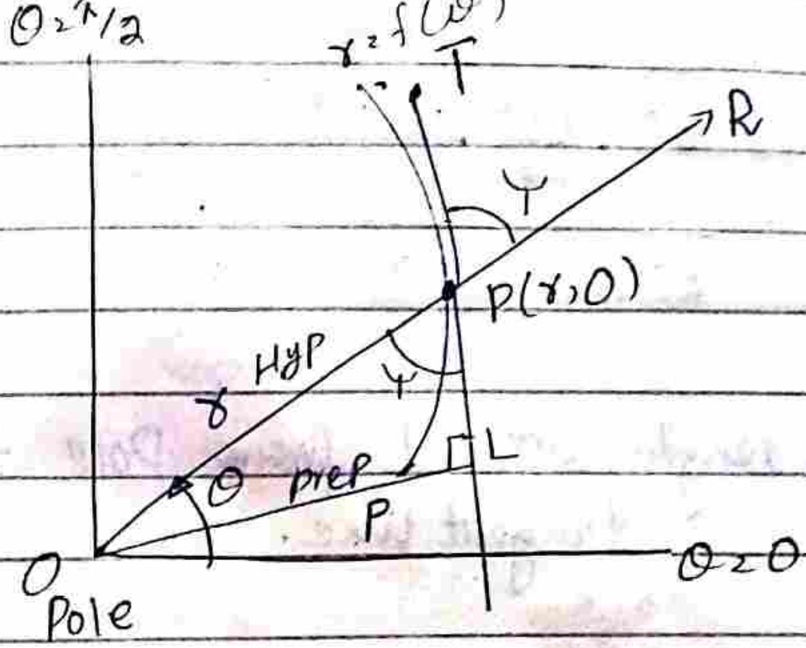
$\psi_1 = \frac{\pi}{4}$, $\psi_2 = -\pi/4$

Angle b/w given $= |\psi_1 - \psi_2|$

$= \left|\frac{\pi}{4} + \pi/4\right|$

$= \pi/2$

2. Length of Perpendicular from pole to the tangent line, Pedal equation:-



from right angled triangle ΔOLP

$$\frac{p}{r} = \sin\psi$$

$$\boxed{p = r\sin\psi}$$

By taking Square on both sides

$$p^2 = r^2 \sin^2\psi$$

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\psi}$$

$$\frac{1}{p^2} = \frac{\cos^2\psi + \sin^2\psi}{r^2\sin^2\psi}$$

$$= \frac{\cos^2\psi}{r^2\sin^2\psi} + \frac{\sin^2\psi}{r^2\sin^2\psi}$$

$$= \frac{1}{\frac{r^2\sin^2\psi}{\cos^2\psi}} + \frac{1}{r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2 \tan^2\psi} + \frac{1}{r^2}$$

$$= \frac{1}{r^2\left(\frac{r\,d\theta}{dr}\right)^2} + \frac{1}{r^2}$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

P = length of $\perp$ from Pole to the tangent line.

θ = angle between R and $\theta = 0$

ψ = angle b/w R and tangent line.

α = angle b/w Tangent line and $\theta = 0$

Pedal Equation:-

The relation between "p" and "r" is called the Pedal Equation.

Exp :- $p = 3r$

Example of Pedal Equation

Find the Pedal equation of
$r^m = a^m \cos m\theta$.

$r^m = a^m \cos m\theta$

$m r^{m-1} \frac{dr}{d\theta} = -m a^m \sin\theta$

$\frac{dr}{d\theta} = -\frac{a^m \sin\theta}{r^{m-1}}$

Consider :-

$\frac{r d\theta}{dr} = r \cdot \left(-\frac{r^{m-1}}{a^m \sin\theta} \right)$

$= -\frac{r^{m-1+1}}{a^m \sin m\theta}$

$\tan\psi = -\frac{a^m \cos m\theta}{a^m \sin m\theta}$

$= -\cot m\theta$

$\tan\psi = \tan\left(\frac{\pi}{2} + m\theta\right) \quad \because \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\psi = \pi/2 + m\theta$

using $P = r \sin\psi$

$P = r \sin\left(\frac{\pi}{2} + m\theta\right)$

$P = r \cos m\theta \quad \because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

$p = r \cdot \frac{r^m}{a^m}$ $\because r^m = a^m \cos m\theta$

$\boxed{a^m p = r^{m+1}}$ This is a require pedal equation.

Mid term

GRAND QUIZ.

Best of Luck.

Notes prepared by:-

* Remember me in your prayers.
Thanks. ☺