Lecture 23

Maximum & Minimum values of function:-

Absolute Maximum:-

If $f(x_0) \geq f(x)$ for all x in the domain of f, Then $f(x_0)$ is called absolute maximum value or simply maximum value of f.

Absolute Minimum:-

If $f(x_0) \leq f(x)$ for all x in the domain of f Then $f(x_0)$ is called the absolute minimum or simply minimum value of f.

Remark:-

Absolute extremum is the max or min value over the entire domain of f.

Example:-

Find max & min value of $f(x) = 2x+1$ over $[0, 3)$.

put $x = 0$

$f(0) = 2(0)+1 = 1$

put $x = 1$

$f(1) = 2(1)+1 = 3$

put $x = 2$

$f(2) = 2(2)+1 = 4+1 = 5$

So minimum (absolute) value is 1 at $x=0$ & function has no ~~max~~ absolute max value.

Extreme value Theorem:-

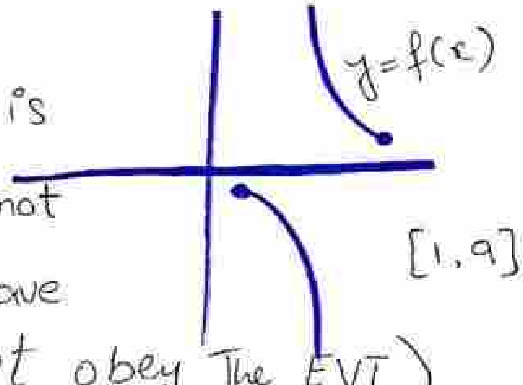
If a function is continuous on a closed interval $[a, b]$, Then f has both a maximum value & a minimum value on $[a, b]$.

Example:-

In previous example $f(x) = 2x+1$, function is continuous but has no maximum value, so it does not obey Extreme value Theorem bcz interval was not closed.

Example:-

In This graph interval is closed but function is not continuous, so does not have max or min value (doesn't obey The EVT).



Theorem:-

If a function f has an extreme value (min/max) on an open interval (a, b), then the extreme value occurs at a critical point of f.

Steps:-

1- Find critical points of f in (a,b).

2- Evaluate f at all critical points & end points

3- The largest of the values is maximum value.

4- The smallest of the values is minimum value.

Example:- Find maximum & minimum value of $f(x) = 2x^3 - 15x^2 + 36x$ on interval $[1,5]$.

Step 01 Take derivative to find critical points.

$$f'(x) = 2(3x^2) - 15(2x) + 36(1)$$
$$= 6x^2 - 30x + 36$$

now put $f'(x) = 0$

$$6x^2 - 30x + 36 = 0$$
$$6(x^2 - 5x + 6) = 0$$
$$x^2 - 5x + 6 = 0$$
$$x^2 - (3+2)x + 6 = 0$$
$$x^2 - 3x - 2x + 6 = 0$$
$$x(x-3) - 2(x-3) = 0$$
$$(x-2)(x-3) = 0$$
$$x - 2 = 0 \quad / \quad x - 3 = 0$$
$$\boxed{x = 2} \quad / \quad \boxed{x = 3}$$

Step 02:-

put $x = 2$ in $f(x)$

$$f(x) = 2x^3 - 15x^2 + 36x$$
$$f(2) = 2(2)^3 - 15(2)^2 + 36(2) \Rightarrow 2(8) - 15(4) + 72$$
$$= 16 - 60 + 72 \Rightarrow \boxed{28}$$

put $x = 3$ in $f(x)$

$f(x) = 2x^3 - 15x^2 + 36x$

$f(3) = 2(3)^3 - 15(3)^2 + 36(3)$

$= 2(27) - 15(9) + 108$

$= 54 - 135 + 108$

$\boxed{f(3) = 27}$

put $x = 1$

$f(1) = 2(1)^3 - 15(1)^2 + 36(1)$

$= 2 - 15 + 36$

$\boxed{f(1) = 23}$

put $x = 5$

$f(5) = 2(5)^3 - 15(5)^2 + 36(5)$

$= 2(125) - 15(25) + 180$

$= 250 - 375 + 180$

$\boxed{f(5) = 55}$

Step 4:
So $f(x)$ has maximum value at $x = 5$ (i-e) 55.

Step 5 $f(x)$ has minimum value 23 at $x = 3$.

Some Results:-

$\lim_{x \to -\infty} f(x) = +\infty$
$\lim_{x \to +\infty} f(x) = +\infty$
} $f(x)$ has a ~~maxi~~ minimum but no max value.

$\lim_{x \to -\infty} f(x) = -\infty$
$\lim_{x \to -\infty} f(x) = -\infty$
} $f(x)$ has a max but no min value.

$\lim_{x \to -\infty} f(x) = -\infty$
$\lim_{x \to +\infty} f(x) = +\infty$
$\lim_{x \to -\infty} f(x) = +\infty$
$\lim_{x \to +\infty} f(x) = -\infty$
} $f(x)$ has neither max nor min value.

Same for $\lim_{x \to a}$ & $\lim_{x \to b}$

Example:-

Find max & min value of $f(x)=x^4+2x^3-1$ on interval $(-\infty,+\infty)$

$$f'(x)=4x^3+6x^2$$

put $f'(x)=0$.

$$4x^3+6x^2=0$$
$$2x^2(2x+3)=0$$
$$2x^2=0 \quad , \quad 2x+3=0$$
$$x^2=0 \quad , \quad 2x=-3$$
$$\boxed{x=0} \quad , \quad \boxed{x=\frac{-3}{2}}$$

put $x=0$ in $f(x)$

$$f(x)=x^4+2x^3-1$$
$$=(0)^4+2(0)^3-1$$
$$=0+0-1$$
$$f(0)=-1$$

put $x=\frac{-3}{2}$

$$f\left(\frac{-3}{2}\right)=\left(-\frac{3}{2}\right)^4+2\left(\frac{-3}{2}\right)^3-1$$
$$=\frac{81}{16}-\frac{54}{8}-\frac{1}{1}$$
$$=\frac{81-108-16}{16}$$
$$f\left(\frac{-3}{2}\right)=-\frac{43}{16}$$

put $x=+\infty$

$$\lim_{x\to+\infty}(x^4+2x^3-1)$$
$$=+\infty$$

put $x=-\infty$

$$\lim_{x\to-\infty}(x^4+2x^3-1)$$
$$=+\infty$$

So $f(x)$ has no max value.

$f(x)$ has min value at $x=\frac{-3}{2}$ (i-e) $-\frac{43}{16}$.

Applied minimum & maximum problems:-

Example:-

Find dimension of a rectangle with perimeter 100 ft whose area is as large as possible.

let $x =$ length, $y =$ width, $A =$ area

$P = 2(x+y)$

$100 = 2(x+y)$

$50 = x+y$

$50 - x = y$

$A = xy$

put $y = 50 - x$

$A = x(50-x)$

$= 50x - x^2$

put $50x - x^2 = 0$

$x(50-x) = 0$

$x = 0$ or $50 - x = 0$. So end points are $[0, 50]$

Now find critical points

$A'(x) = 0$

$50(1) - 2x = 0$

$50 - 2x = 0$

$50 = 2x$

$\boxed{25 = x}$

put $x = 25$ in

$A = 50x - x^2 \Rightarrow 50(25) - (25)^2 \Rightarrow 1250 - 625 \Rightarrow 625$

put $x = 0$

$A = 50(0) - 0 \Rightarrow 0$

put $x = 50$

$A = 50(50) - (50)^2 \Rightarrow 2500 - 2500 \Rightarrow 0$

So max value of A is 625 at $x = 25$

$x - 15 = 0$ | $x - 8 = 0$

$x = 15$ | $\boxed{x = 8}$

x can't be 15 bcz it makes ~~...~~ length negative. So end points are $[0, 8]$

Now we'll find critical points.

$$V = 480x - 92x^2 + 4x^3$$

$$\frac{dV}{dx} = 480 - 184x + 12x^2$$

put $\frac{dV}{dx} = 0$

$$480 - 184x + 12x^2 = 0$$

$$4(120 - 46x + 3x^2) = 0$$

$$3x^2 - 46x + 120 = 0$$

$$3x^2 - (36 + 10)x + 120 = 0$$

$$3x^2 - 36x - 10x + 120 = 0$$

$$3x(x - 12) - 10(x - 12) = 0$$

$$(x - 12)(3x - 10) = 0$$

$x - 12 = 0$ | $3x - 10 = 0$

$x = 12$ | $3x = 10$

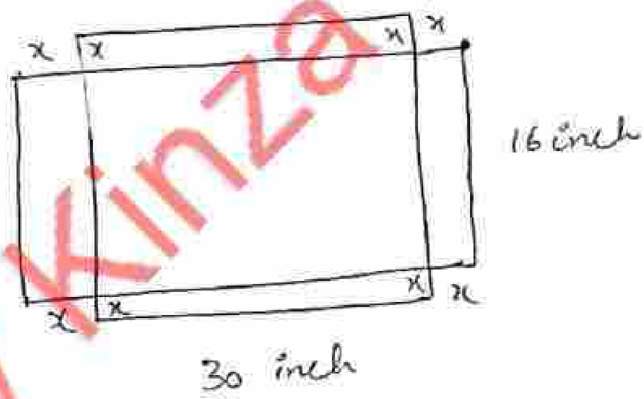
$\boxed{x = \frac{10}{3}}$

ignore $x = 12$ bcz it is out of $[0, 8]$.

Now use values of $x\ (0, 8, \frac{10}{3})$ to find maximum value of V.

Example:-

An open box to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the 4 corners & bending up the sides. What size should the squares be to obtain a box with largest possible volume?



Let x = length of sides to be cut

V = Volume of resulting box.

dimensions will be

$(16-2x)$ by $(30-2x)$

As $V = (\text{length})(\text{width})(\text{height})$

$$V = (16-2x)(30-2x)(x)$$

$$= 480x - 92x^2 + 4x^3$$

put $V=0$ to find end points.

$$480x - 92x^2 + 4x^3 = 0$$

$$4x(120 - 23x + x^2) = 0$$

$$4x = 0 \quad , \quad x^2 - 23x + 120 = 0$$

$$\boxed{x=0} \quad , \quad x^2 - (15+8)x + 120 = 0$$

$$x^2 - 15x - 8x + 120 = 0$$

$$x(x-15) - 8(x-15) = 0$$

$$(x-15)(x-8) = 0$$

put $x=0$

$V = 480x - 92x^2 + 4x^3$

$= 480(0) - 92(0)^2 + 4(0)^3$

$V = 0$

put $x = 8$

$V = 480x - 92x^2 + 4x^3$

$V = 480(8) - 92(8)^2 + 4(8)^3$

$V = 3840 - 5888 + 2048$

$V = 0$

put $x = \frac{10}{3}$

$V = 480\left(\frac{10}{3}\right) - 92\left(\frac{10}{3}\right)^2 + 4\left(\frac{10}{3}\right)^3$

$= \frac{4800}{3} - \frac{9200}{9} + \frac{4000}{27}$

$= \frac{43200 - 27600 + 4000}{27}$

$\boxed{V = \frac{19600}{27}}$

So function of volume has its maximum value $\frac{19600}{27}$ at $x = \frac{10}{3}$.

Lecture 24

Newton's Method:-
Rolle's Theorem:-
Mean value Theorem:-

⇒ Newton's Method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(i-e) if $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

if $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

& so on.

Example:- The equation $x = \cos x$ has a solution between 0 and 1. Approximate it using newton's method. $f(x) = x - \cos x$

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1+\sin(x_n)}$$

put $x_1 = 1$

put $n=1$, $x_{1+1} = x_1 - \frac{x_1 - \cos(x_1)}{1+\sin(x_1)}$

D AA .

$$x_2 = 1 - \frac{1-\cos(1)}{1+\sin(1)} = 0.7503$$

put $n=2$

$$x_{2+1} = x_2 - \frac{x_2 - \cos(x_2)}{1+\sin(x_2)}$$

$$x_3 = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1+\sin(0.7503)}$$

$$x_3 = 0.7391$$

You may continue, but approximate solution is 0.7391.

Some difficulties with Newton's Method.

⇒ It does not always work.

⇒ If for some values of n . $f'(x_n)$ Then formula involves division by zero & we are out of business.

⇒ Such a ~~slope~~ case will occur if the Tangent line for some some approximation has slope zero or is parallel to x-axis.

⇒ Sometimes approximation don't converge to a solution.

For example in $x^{\frac{1}{3}} = 0$

$$x_{n+1} = x_n - \frac{(x_n^{\frac{1}{3}})}{\frac{1}{3}(x_n)^{-\frac{2}{3}}} \Rightarrow x_n - 3(x_n)^{\frac{1}{3}}.(x_n)^{\frac{2}{3}} \Rightarrow x_n - 3(x_n)^{\frac{1}{3}+\frac{2}{3}}$$

$$= x_n - 3x_n = -2x_n$$

let $x_1 = 1$ & Then follow approximation, value do not converge

Rolle's Theorem:-

Let f be differentiable on (a,b) & continuous on $[a,b]$. If $f(a)=f(b)=0$ then there is at least one point c in (a,b) where $f'(c)=0$.

Example:- $f(x) = \sin x$ $[0, 2\pi]$

As $\sin x$ is differentiable on $(0, 2\pi)$ & continuous on $[0, 2\pi]$. & $\sin(0) = \sin(2\pi) = 0$

So $f'(c) = \cos x = 0$ at any number in 0 & 2π

Here the value of c is $\frac{\pi}{2}$ or $90°$.

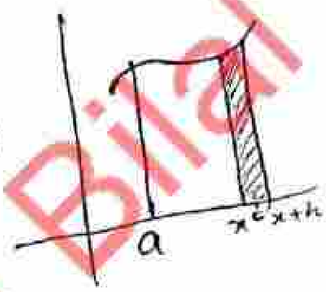
Mean value Theorem:-

Let f be differentiable on (a,b) & continuous on $[a,b]$. Then there is at least one point c in (a,b) where $f'(c) = \frac{f(b)-f(a)}{b-a}$

Lecture 25 Integration

Area problem

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} f(c)$$



As h goes to zero, c approaches x.
$f(c)$ goes to $f(x)$ as c goes to x.

Antiderivative:-

If $F(x)$ is any antiderivative of $f(x)$ on a given interval, then for any value of c, the function $F(x)+C$ is also an antiderivative of $f(x)$ on that interval, moreover every antiderivative of $f(x)$ on the interval is expressible in the form $F(x)+C$, where C is constant.

Indefinite integral:-

The process of finding anti derivative is called anti-differentiation or Integration.

function in the form $F(x)+C$ are called anti derivative of $f(x)$. **For example** $\frac{1}{3}x^3$, $\frac{1}{3}x^3 - \pi$, $\frac{1}{3}x^3 + C$ are all anti-derivatives of $f(x) = x^2$.

We denote anti-derivative by $\int f(x)dx = F(x) + c$

The symbol $\int$ is called integral sign & $f(x)$ is called integrand. It is read as "Indefinite integral of $f(x)$ equals $F(x)$"

C is called constant of integration

Example:- $\int f(x)\,dx = ?$ when $f(x) = x^2$

$$\int f(x)\,dx = \int x^2\,dx$$
$$= \frac{x^{2+1}}{2+1} + c \Rightarrow \frac{x^3}{3} + c$$

Example:-

$$\int x^2\,dx = \frac{x^3}{3} + c$$

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + c \Rightarrow \frac{x^4}{4} + c$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx \Rightarrow \frac{x^{-5+1}}{-5+1} + c \Rightarrow \frac{x^{-4}}{-4} + c \Rightarrow \frac{-1}{4x^4} + c$$

Derivative Formula	Integration Formula
$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$	$\int 3x^2\,dx = 3\frac{x^{2+1}}{2+1} \Rightarrow \frac{3x^3}{3} \Rightarrow x^3 + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dt}(\tan t) = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + c$
$\frac{d}{du}(u^{\frac{3}{2}}) = \frac{3}{2}u^{\frac{3}{2}-1} \Rightarrow \frac{3}{2}u^{\frac{1}{2}}$	$\int \frac{3}{2}u^{\frac{1}{2}}\,du = \frac{3}{2}\left(\frac{u^{\frac{1}{2}+1}}{\frac{3}{2}}\right) = u^{\frac{3}{2}} + c$

Properties of Definite Integral:-

⇒ constant can be moved through an integral sign.

$$\int c\, f(x)\,dx = c\int f(x)\,dx$$

⇒ Anti-derivative of a sum is the sum of anti-derivatives. $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

⇒ Anti-derivative of a difference is the difference of the anti-derivatives.

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Example:- Evaluate $\int 4\cos x\,dx$

$$\int 4\cos x\,dx = 4\int \cos x\,dx = 4[\sin x + c]$$

$$= 4\sin x + 4c = 4\sin x + k.$$

Example:- $\int (x^2+x)\,dx$

$$= \int x^2\,dx + \int x\,dx = \frac{x^{2+1}}{3} + \frac{x^{1+1}}{2} + c$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + c$$

Generalized Version:-

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots\cdots + c_n f_n(x)]\,dx$$

$$= c_1\int f_1(x)\,dx + c_2\int f_2(x)\,dx + \cdots + c_n\int f_n(x)\,dx$$

Example:- $\int (3x^6 - 2x^2 + 7x + 1)\,dx$

$$= 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + 1\int \cdot dx$$

$$= 3\frac{x^7}{7} - 2\frac{x^3}{3} + \frac{7x^2}{2} + x + c$$

Example:- $\int \frac{\cos x}{\sin^2 x}\,dx$

$$= \int \frac{\cos x}{\sin x . \sin x}\,dx$$

$$= \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}\,dx$$

$$= \int (\operatorname{cosec} x \cot x)\,dx$$

$$= -\operatorname{cosec} x + c$$

Lecture 26:- Integration by Substitution:-

This is like chain rule of differentiation.
Integration by Substitution is used for solving composit functions.

$$\frac{d}{dx}(G(u)) = \frac{d}{du}(G(u)) \cdot \frac{du}{dx} = f(u)\frac{du}{dx}$$

Take integral

$$\int\left(f(u)\frac{du}{dx}\right)dx = G(u)+C.$$

$$\Rightarrow \int\left(f(u)\frac{du}{dx}\right)dx = \int f(u)\,du$$

Example:- $\int (x^2+1)^{50} \cdot 2x\, dx$

Let $u = (x^2+1)$
Take derivative

$$\frac{du}{dx} = 2x+0 \Rightarrow 2x$$

$$du = 2x\, dx$$

$$\int u^{50}\, du$$

$$= \frac{u^{50+1}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C.$$

Don't just add 1 to the origional problem, it'll be incorrect.

Steps for u-substitution:-

1. Make choice for u, say $u = g(x)$.
2. Compute $\frac{du}{dx} = g'(x)$
3. Make substitutions $u = g(x)$ & $du = g'(x)\,dx$.
   (Here no x will be in problem, whole function'll be in the form of u.)
4. Evaluate the resulting integral.
5. Replace u by $g(x)$ so the final answer is in x.

Differentiation formula	Integration formula
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = x + c$
$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = x^r$	$\int x^r dx = \frac{x^{r+1}}{r+1} + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + c$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\frac{d}{dx}(-\operatorname{cosec} x) = \cancel{\csc x \tan} \;\; \csc x \cot x$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

Example:-

$$\int (x-8)^{23} dx$$

let $u = x - 8$

$$\frac{du}{dx} = 1$$

$$du = 1\,dx$$

$$du = dx$$

$$\int u^{23} \, dx$$

$$= \frac{u^{23+1}}{24} + c$$

$$= \frac{(x-8)^{24}}{24} + c$$

Example:-

$$\int \cos(5x)\,dx$$

let $u = 5x$

$$\frac{du}{dx} = 5$$

$$du = 5\,dx$$

$$\frac{du}{5} = dx$$

$$\int \cos u \cdot \frac{du}{5}$$

$$= \frac{1}{5}\int \cos u \cdot du$$

$$= \frac{1}{5} \sin u + c$$

$$= \frac{1}{5}\sin(5x) + c$$

Example:- $\int \sin^2 x \cos x \, dx$

let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x \, dx$$

$$\int u^2 \cdot du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{(\sin x)^3}{3} + c$$

$$= \frac{\sin^3 x}{3} + c$$

Example:- $\int \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$

let $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2}(x)^{\frac{1}{2}-1}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}}$$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$2du = \frac{1}{\sqrt{x}} dx$$

$$\int \cos\sqrt{x} \cdot \frac{1}{\sqrt{x}} dx$$

$$= \int \cos u \cdot 2du$$

$$= 2\int \cos u \cdot du$$

$$= 2 \sin u + c$$

$$= 2\sin(\sqrt{x}) + c.$$

Example:- $\int t^4\sqrt{3-5t^5}\,dt$

let
$u = 3-5t^5$

$\frac{du}{dt} = 0-5(5t^4)$

$du = -25t^4\,dt$

$\frac{du}{-25} = t^4\,dt$

$\int \sqrt{3-5t^5}\; t^4 dt$

$= \int \sqrt{u} \cdot \frac{du}{-25}$

$= \frac{-1}{25}\int \sqrt{u}\,du$

$= \frac{-1}{25}\; \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{-1}{25}\cdot \frac{u^{3/2}}{3/2} + c$

$= \frac{-1}{25} \times \frac{2}{3} \times u^{3/2} + c$

$= \frac{-2(3-5t^5)^{3/2}}{75} + c$

if $\int t^4\sqrt[3]{3-5t^4}\,dt$

$= \int \sqrt[3]{3-5t^4}\; t^4 dt$

$= \int \sqrt[3]{u}\cdot \frac{du}{-\cancel{1}25}$

$= \frac{-1}{25}\int \sqrt[3]{u}\,du$

$= -\frac{1}{25}\cdot \frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c$

$= -\frac{1}{25}\cdot \frac{u^{\frac{4}{3}}}{4/3} + c$

$= -\frac{1}{25}\cdot\frac{3}{4}\cdot u^{\frac{4}{3}} + c$

$= \frac{-1}{25}\cdot\frac{3}{4}\cdot(3-5t^4)^{\frac{4}{3}} + c$

$= \frac{-3}{100}(3-5t^4)^{\frac{4}{3}} + c$

Lecture 27:- Sigma Notation:-

⇒ used to write lengthy sums in compact Form.

⇒ Sigma (Σ) is an upper case letter in Greek.

⇒ Σ is called Sigma / Summation.

Examples:-

i) $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

ii) $\sum_{k=4}^{8} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

iii) $\sum_{k=1}^{5} 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5)$

$= 2 + 4 + 6 + 8 + 10$

iv) $\sum_{k=0}^{5} (-1)^k(2k+1) = (-1)^0(2(0)+1) + (-1)^1[2(1)+1] + (-1)^2[2(2)+1] +$

$(-1)^3(2(3)+1) + (-1)^4[2(4)+1] + (-1)^5[2(5)+1]$

$= 1 - 3 + 5 - 7 + 9 - 11$

In $\sum_{k=0}^{5} (2k)$ The number on the top (5) is called upper limit of summation & the number on the bottom (0) is called lower limit.

k is called index of the Summation.

any letter can be used as index instead k.

Example:- $\sum_{i=1}^{4} \frac{1}{i}$, $\sum_{n=1}^{4} \frac{1}{n}$, $\sum_{j=1}^{4} \frac{1}{j}$ all are same.

(i-e) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

- If upper & lower limits are same then the summation reduces to just one term.

$$\sum_{k=2}^{2} k^3 = 2^3$$

- If in the summation, expression on right of summation does not involve the index, then

$$\sum_{i=1}^{5} 2 = 2+2+2+2+2$$

$$\sum_{k=3}^{6} x^3 = x^3 + x^3 + x^3 + x^3$$

- A sum can be written in more than one notations if we change the limits.

$$\sum_{k=1}^{5} 2k = 2+4+6+8+10$$

$$\sum_{k=0}^{4} (2k+2) = 2+4+6+8+10$$

$$\sum_{k=2}^{6} (2k-2) = 2+4+6+8+10$$

Changing the index of summation:-

Example:- Express $\sum_{k=3}^{7} 5^{k-2}$ in sigma notation so that lower limit is 0 rather than 3.

Define a new index j by

Take $k = 3$

$k - 3 = 0$

$j = 0$

$\left(\text{say } j = k-3 \right)$

Then $j + 3 = k$

Put $k = 3$
$j = 3 - 3 = 0$

Put $k = 7$
$j = 7 - 3 = 4$

Put in

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} 5^{(j+3)-2}$$

$$= \sum_{j=0}^{4} 5^{j+1}$$

- To represent a general sum, we'll use letters with subscripts.

Example:- $a_1 + a_2 + a_3$ represent general sum with three terms. can also be written as

$$\sum_{k=1}^{3} a_k.$$

- general sum with n terms can be written as

$$\sum_{k=1}^{n} b_k = b_1 + b_2 + b_3 + \cdots + b_n.$$

Properties of Sigma Notation:-

i) $\displaystyle\sum_{k=1}^{n} c\,a_k = c \sum_{k=1}^{n} a_k$

ii) $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

iii) $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Theorems:-

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^3$$

Example:- $\sum_{k=1}^{30} k(k+1)$

$$= \sum_{k=1}^{30} (k^2+k) \Rightarrow \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{30(30+1)[2(30)+1]}{6} + \frac{30(30+1)}{2}$$

$$= \frac{(30)(31)(61)}{6} + \frac{(30)(31)}{2}$$

$$= \frac{56730}{6} + \frac{930}{2}$$

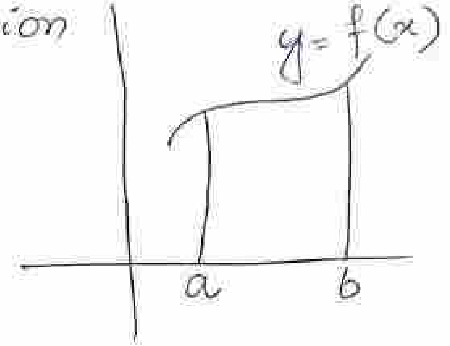
$$= 9455 + 465$$

$$= 9920$$

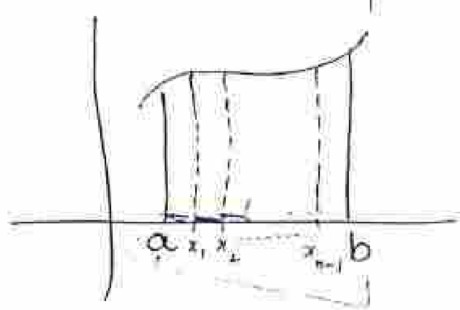
- In above Three Theorems (~~like~~, left part is called open form (like in 1st, $1+2+3+\cdots+n$) & right part is called closed ~~part~~ form (like $\frac{n(n+1)}{2}$).

Lecture 28 Area as Limits.

In This figure, There is a region bounded below by x-axis, on The sides by The lines $x=a$ & $x=b$, Above by a curve of continuous function $f(x)=y$ which is also non-negative on interval $[a,b]$. Such an area can be calculated using anti-derivatives.



We can divide This graph/area/region into n equal parts of width $\frac{b-a}{n}$ by inserting $n-1$ equally spaced points between a & b $(x_1, x_2, x_3, \ldots, x_n)$.



$$\Delta x = \frac{b-a}{n}$$

$$\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

Left End Point:-

$$x_k^* = x_{k-1} = a + (k-1)\Delta x$$

Right End Point:-

$$x_k^* = x_k = a + k\Delta x$$

Mid Point:-

$$x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right)\Delta x$$

Example:- Use defination of area with x_k^* as right end point of each subinterval to find area under line $y = x$ over $[1, 2]$
$[a, b]$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

For right end point

$$x_k^* = a + k\Delta x$$

here $a = 1$ & $\Delta x = \frac{1}{n}$

$$x_k^* = 1 + k\left(\frac{1}{n}\right)$$

$$= 1 + \frac{k}{n}$$

$$f(x_k^*)\Delta x = x_k^* \Delta x$$

$$= \left(1 + \frac{k}{n}\right)\left(\frac{1}{n}\right)$$

$$= \left(\frac{1}{n} + \frac{k}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k}{n^2} \right)$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{n} + \sum_{k=1}^{n} \frac{k}{n^2} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=1}^{n} 1 + \frac{1}{n^2} \sum_{k=1}^{n} k \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \cdot n + \frac{1}{n^2} \cdot \frac{n(n-1)}{2} \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{(n-1)}{2n} \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{n}{2n} - \frac{1}{2n} \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{1}{2} - \frac{1}{2n} \right]$$

$$= 1 + \frac{1}{2} - \frac{1}{2(\infty)}$$

$$= \frac{2+1}{2} - \frac{1}{\infty}$$

$$= \frac{3}{2} - 0 \Rightarrow \boxed{\frac{3}{2}}$$ Ans.

Example:- Same problem as before but with left end point.

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \underline{\frac{1}{n}}$$

Left end point

$$x_k^* = x_{k-1} = a + (k-1)\Delta x$$

here $a = 1$ & $\Delta x = \frac{1}{n}$

$$x_k^* = 1 + (k-1)\frac{1}{n}$$

$$\underline{x_k^* = 1 + \frac{k-1}{n}}$$

now

$$f(x_k^*)\Delta x = x_k^* \Delta x$$

$$= \left(1 + \frac{k-1}{n}\right)\frac{1}{n}$$

$$= \underline{\frac{1}{n} + \frac{k-1}{n^2}}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k-1}{n^2}\right)\left(\quad\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k}{n^2} - \frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{n} + \sum_{k=1}^{n}\frac{k}{n^2} - \sum_{k=1}^{n}\frac{1}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}1 + \frac{1}{n^2}\sum_{k=1}^{n}k - \frac{1}{n^2}\sum_{k=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\cdot n + \frac{1}{n^2}\cdot\frac{n(n-1)}{2} - \frac{1}{n^2}\cdot n\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{n-1}{2n} - \frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{n}{2n} - \frac{1}{2n} - \frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[1 + \frac{1}{2} - \frac{1}{2n} - \frac{1}{n}\right]$$

$$= 1 + \frac{1}{2} - \frac{1}{2(\infty)} - \frac{1}{\infty}$$

$$= \frac{2+1}{2} - \frac{1}{\infty} - 0$$

$$= \frac{3}{2} - 0 - 0$$

$$= \boxed{\frac{3}{2}}$$ Ans.

Example:- Use area defination with right end points of each subinterval to find area under parabola $y = 9 - x^2$ over $[0, 3]$.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

right end point:-

$$x_k^* = a + k\Delta x$$

here $a = 0$ & $\Delta x = \frac{3}{n}$

$$x_k^* = 0 + k\left(\frac{3}{n}\right) = \frac{3k}{n}$$

Now $f(x_k^*)\Delta x = (9 - x_k^{*2})\Delta x$

$$= \left[9 - \left(\frac{3k}{n}\right)^2\right]\frac{3}{n} \Rightarrow \left[9 - \frac{9k^2}{n^2}\right]\frac{3}{n}$$

$$= \frac{27}{n} - \frac{27k^2}{n^3}$$

$$\text{Area} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{27}{n} - \frac{27k^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{27}{n}\sum_{k=1}^{n}1 - \frac{27}{n^3}\sum_{k=1}^{n}k^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n}\cdot n - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{27}{n^2}\,\frac{(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{27(2n^2+3n+1)}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{54n^2+81n+27}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{54n^2}{6n^2} - \frac{81n}{6n^2} - \frac{27}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{54}{6} - \frac{81}{6n} - \frac{27}{6n^2}\right]$$

$$= \left(27 - \frac{54}{6} - \frac{81}{6(\infty)} - \frac{27}{6(\infty)}\right)$$

$$= 27 - 9 - 0 - 0$$

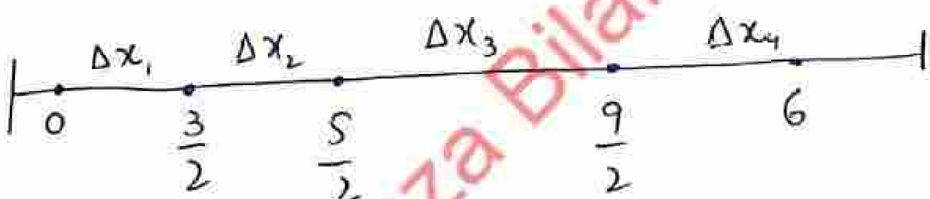
$$= \boxed{18} \text{ Ans.}$$

Lecture 29 The Definite Integral

As the formula for width is $\Delta x = \frac{b-a}{n}$

If the width betwen subintervals is not equal then this formula changes.

If we want to find width of the largest subinterval, then it is called <u>mesh size</u> & represented by <u>$\max \Delta x_k$</u> Read as <u>"maximum of the Δx_k"</u>. For example



$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = \frac{4}{2} = 2$$

Theorem:- Area under curve

If function f is continuous on $[a,b]$ & if $f(x) \geq 0$ for all x in $[a,b]$, then area under curve over $[a,b]$ is $A = \lim\limits_{\max \Delta x_k \to 0} \sum\limits_{k=1}^{n} f(x_k^*) \Delta x_k$

can also be written as $\int_a^b f(x)\,dx$

$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is called Riemann Sum.

$\int_a^b f(x)\,dx$ is called definite integral of f from a to b & b, a are called upper & lower limits respectively.

Example:- Evaluate $\int_2^4 (x-1)\,dx$

This represents area under curve $y = x-1$ over $[2,4]$.

$$\int_2^4 x\,dx - \int_2^4 1\,dx$$

$$= \left|\frac{x^2}{2}\right|_2^4 - \left| x \right|_2^4 + c$$

$$= \left(\frac{4^2}{2} - \frac{2^2}{2}\right) - (4-2)$$

$$= \left(\frac{16}{2} - \frac{4}{2}\right) - 2$$

$$= (8-2) - 2$$

$$= 6 - 2$$

$$= 4 \quad \text{Ans.}$$

Example:- $\int_2^4 (1-x)\,dx$

$$= \int_2^4 1\,dx - \int_2^4 x\,dx$$

$$= \left| x \right|_2^4 - \left| \frac{x^2}{2} \right|_2^4$$

$$= (4-2) - \left(\frac{4^2}{2} - \frac{2^2}{2} \right)$$

$$= 2 - \frac{16}{2} + \frac{4}{2}$$

$$= 2 - 8 + 2$$

$$= -4 \quad \text{Ans.}$$

-ve sign just shows that area below x-axis is larger than above.

Defination:-

- If a is in the domain of f.

$$\int_a^a f(x)\,dx = 0$$

- If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Properties of definite integral

- $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$
- $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
- $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

Theorem:- If f is integrable on a closed interval containing three points a, b & c then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Example:- $\int_1^5 f(x)\, dx = -1$, $\int_3^5 f(x)\, dx = 3$

$\int_3^5 g(x)\, dx = 4$, $\int_1^3 f(x)\, dx = ?$

$$\int_1^5 f(x)\, dx = \int_1^3 f(x)\, dx + \int_3^5 f(x)\, dx$$

$$-1 = \int_1^3 f(x)\, dx + 3$$

$$-4 = \int_1^3 f(x)\, dx$$

Theorem:-

If f is integrable on $[a,b]$ & $f(x) \geq 0$ for all x in $[a,b]$ Then $\int_a^b f(x) \geq 0$

If f & g are integrable functions on $[a,b]$ & $f(x) \geq g(x)$ for all x in $[a,b]$ Then $\int_a^b f(x) \geq \int_a^b g(x)$

works with $>, <, \leq$ also.

Example:- Show That $\int_0^1 \frac{\cos x}{2x^3-5} dx$ is negative.

On interval $[0,1]$, $\cos x > 0$ but $2x^3-5 < 0$, So $f(x)$ as a whole is < 0

So integral is negative.

Defination:- Function is said to be bounded on $[a,b]$ if There is a +ve number M such That

$$-M \leq f(x) \leq M$$

so $y = -M$ & $y = M$

Lecture 30

First Fundamental Theorem of Calculus.

Theorem:- If f is continuous on $[a,b]$ & if F is an antiderivative of f on $[a,b]$, Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

$$= \int_a^b f(x)\,dx$$

or

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$

Example:- Evaluate $\int_1^2 x\,dx$

$$= \frac{x^2}{2}\Big]_1^2 \Rightarrow \frac{(2)^2}{2} - \frac{(1)^2}{2}$$

$$= \frac{4}{2} - \frac{1}{2} \Rightarrow \frac{4-1}{2}$$

$$= \frac{3}{2} \quad \underline{\text{Ans}}.$$

Properties

$$\left[c\,F(x)\right]_a^b = c\left[F(x)\right]_a^b$$

$$\left[F(x)+G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$$

$$\left[F(x)-G(x)\right]_a^b = \left[F(x)\right]_a^b - \left[G(x)\right]_a^b$$

As

$$\int f(x)\,dx = F(x)+C$$

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

Example:- Find area under curve $y=\cos x$ over $[0, \pi]$.

since $\cos x \geq 0$ for $0 \leq x \leq \frac{\pi}{2}$

$$A = \int_0^{\pi/2} \cos x\,dx$$

$$= \left|\sin x\right|_0^{\pi/2} \Rightarrow \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1-0 \Rightarrow 1 \text{ Ans.}$$

Example:- Evaluate $\int_0^3 (x^3-4x+1)\,dx$

$$= \int_0^3 x^3\,dx - \int_0^3 4x\,dx + \int_0^3 1\,dx$$

$$= \left|\frac{x^4}{4}\right|_0^3 - \left|\frac{4x^2}{2}\right|_0^3 + |x|_0^3$$

or $$\left|\frac{x^4}{4} - \frac{4x^2}{2} + x\right|_0^3$$

$$= \left(\frac{(3)^4}{4} - 2(3)^2 + 3\right) - \left(\frac{0^4}{4} - \frac{4(0)}{2} + 0\right)$$

$$= \frac{81}{4} - 18 + 3 \Rightarrow \frac{21}{4}$$ Ans.

Example:- Evaluate $\int_0^6 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2 & x < 2 \\ 3x-2 & x \geq 2 \end{cases}$$

$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

$$= \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$$

$$= \left(\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^6$$

$$= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) + \left(\frac{3}{2}(6^2 - 2^2) - 2(6-2)\right)$$

$$= \frac{8}{3} + \frac{3}{2}(32) - 2(4)$$

$$= \frac{8}{3} + 48 - 6 \Rightarrow \frac{8}{3} + 42 \Rightarrow \frac{128}{3}$$

Mean Value Theorem:-

$$\int_a^b f(x)dx = f(x^*)(b-a)$$

if f is continuous on a closed interval $[a,b]$, Then There is at least one number x^* in $[a,b]$. &

Proof:- $m \le f(x) \le M$

if f assumes a max M & a min m on $[a,b]$

$$m \le f(x) \le M$$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$m\,|x|_a^b \le \int_a^b f(x)\,dx \le M(x)|_a^b$$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$m \le \int_a^b f(x)dx \cdot \frac{1}{b-a} \le M$$

Central value is a number b/w m & M. So by intermediate Theorem f assume $\frac{1}{b-a}\int_a^b f(x)$ on $[a,b]$ for some point $f(x^*)$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(x^*)$$

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

Example:- $f(x)=x^2$ is continuous on $[1,4]$ The MVT for integral guarantees That There exist a number x^* Such That

$$\int_1^4 x^2\,dx = f(x^*)(4-1)$$

$$\left.\frac{x^3}{3}\right|_1^4 = 3(x^*)^2$$

$$\frac{4^3}{3} - \frac{1^3}{3} = 3(x^*)^2$$

$$\frac{64-1}{3} = 3(x^*)^2$$

$$\frac{63}{3\times 3} = x^{*2}$$

7 21

$$\sqrt{7} = x^*$$

Average Value Theorem:-

If f is integrable on $[a,b]$ then the average value (or mean value) of f on $[a,b]$ is defined to be

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

if $y = f(x)$ then f_{avg} is also called average value of y w.r.t x over $[a,b]$.

Lecture 31

Evaluating definite integral by Substitution.

1) Evaluate definite integral by Substitution Then use $\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$

2) First represent definite integral in form

$$\int_a^b f(x)\,dx = \int_a^b h(g(x))\,g'(x)\,dx$$

, Then make substitution $u = g(x)$, $du = g'(x)\,dx$, but now we have to change limits from x to u-limits.

$$u = g(a) \text{ if } x = a$$
$$u = g(b) \text{ if } x = b$$

$$\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} h(u)\,du$$

Example:-

1) $\int_0^2 x(x^2+1)^3\,dx$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$, $\frac{du}{2} = x\,dx$

$\int_0^2 u^3 \frac{du}{2} = \frac{1}{2}\int_0^2 u^3\,du$

$= \frac{1}{2}\left[\frac{u^4}{4}\right]_0^2 \Rightarrow \frac{1}{8}\left[x^2+1\right]_0^{2\;4} \Rightarrow$

$= \frac{1}{8}\left[(2^2+1)-(0^2+1)\right]^4 = \frac{1}{8}(4+1-1]^4$

$= \frac{1}{8}(4)^4 = 78$

OR **Method 2.**

$u = x^2+1$

put $x=0$ | put $x=2$
$u = 0+1$ | $u = 2^2+1$
$u = 1$ | $u = 5$

$\frac{1}{2}\int_1^5 u^3\,du \Rightarrow \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5 \Rightarrow \frac{1}{2}\left[\frac{5^4-1^4}{4}\right]$

$= \frac{624}{8} \Rightarrow 78$ Ans.

Example:- $\int_0^{\pi/4} \cos(\pi - x)\, dx$

let $u = \pi - x$, $du = -1\,dx$
$-du = dx$

put $x = 0$

$u = (\pi - 0)$

$u = \pi$

put $x = \frac{\pi}{4}$

$u = \pi - \frac{\pi}{4}$

$u = \frac{3\pi}{4}$

$$\int_{\pi}^{\frac{3\pi}{4}} \cos u\,(-du) \Rightarrow -\int_{\pi}^{\frac{3\pi}{4}} \cos u\, du$$

$$= -\left[\sin u\right]_{\pi}^{\frac{3\pi}{4}} \Rightarrow -\left[\sin\frac{3\pi}{4} - \sin\pi\right]$$

$$= -\left[\frac{1}{\sqrt{2}} - 0\right] \Rightarrow -\frac{1}{\sqrt{2}} \quad \underline{\text{Ans.}}$$

Approximation by Reimann Sum:-

$$\text{Reimann Sum} = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

used when we find approximate area under curve. If we take limits, it becomes definite integral. If we don't take limit, then it is approximately equal to definite integral.

Lecture 32

Second Fundamental Theorem of Calculus.

Dummy variable:-

If we change letter for variable of integration but don't change limits, then value of definite integral are unchanged.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(y)\,dy = \int_a^b f(u)\,du$$

The letter used for variable is called dummy variable.

Example:-

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right]_1^2 \Rightarrow \frac{2^3}{3} - \frac{1^3}{3} \Rightarrow \frac{8}{3} - \frac{1}{3} \Rightarrow \frac{7}{3}$$

$$\int_1^2 t^2\,dt = \left.\frac{t^3}{3}\right]_1^2 \Rightarrow \frac{2^3}{3} - \frac{1}{3} = \frac{8}{3} - \frac{1}{3} \Rightarrow \frac{7}{3}$$

$$\int_1^2 y^2\,dy = \left.\frac{y^3}{3}\right]_1^2 \Rightarrow \frac{2^3 - 1^3}{3} \Rightarrow \frac{8-1}{3} \Rightarrow \frac{7}{3}$$

Definite integral with variable upper limit:-

In Such integral we will use a different letter for integration for differentiating b/w limit & variable.

Example:- Evaluate $\int_2^x t^2 dt$

$$= \int_2^x t^2 dt \Rightarrow \left.\frac{t^3}{3}\right]_2^x \Rightarrow \frac{x^3}{3} - \frac{2^3}{3} \Rightarrow \frac{x^3}{3} - \frac{8}{3}$$

The Result is a function of x.

Second Fundamental Theorem of Calculus:-

⇒ If integrand is continuous, then the derivative of a definite integral w.r.t its upper limit is equal to integrand evaluated at upper limit.

⇒ The derivative of a function representing the area under curve of another continuous function f is equal to the function f on a given interval.

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

Example:- $f(x) = x^3$ is continuous. Evaluate $\int_1^x t^3 dt$ to check validity of 2nd fundamental Theorem of Calculus.

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \frac{d}{dx}\left(\int_1^x t^3 dt\right)$$

$$= \frac{d}{dx}\left[\frac{t^4}{4}\right]_1^x \Rightarrow \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right]$$

$$= \frac{d}{dx}\left[\frac{x^4-1}{4}\right] \Rightarrow \frac{1}{4}\left[\frac{d}{dx}(x^4-1)\right]$$

$$= \frac{1}{4}\left[4x^3 - 0\right] \Rightarrow \frac{1}{4}(4x^3) \Rightarrow x^3 = f(x)$$

Existence of antiderivative of continuous func

$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ tells us that

$F(x) = \int_a^x f(t)dt$ is an anti-derivative of f on I.

So every continuous function on an interval has an anti-derivative on that interval.

Example:- $\int_1^3 \frac{1}{x} dx$

function is continuous on $[1, 3]$ & is also integrable. By 2nd fundamental Theorem of Calculus. $F(x) = \int_1^x \frac{1}{t} dt$

$$\int_1^x \frac{1}{dt} = F(3) - F(1) = \int_1^3 \frac{1}{t} dt - \int_1^1 \frac{1}{t} dt \Rightarrow \int_1^3 \frac{1}{t} dt$$

$\int_1^x \frac{1}{t} dt$ defines a function in terms of integral

Lecture 33

Application to the definite integral

Area b/w two curves:- Suppose that f & g are continuous functions on interval $[a, b]$ & $f(x) \geq g(x)$. This means $y = f(x)$ is above $g(x)$ & two can touch but not cross.

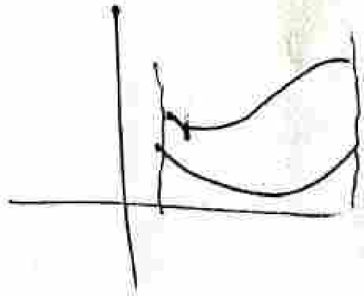
If f & g are non-negative then

$$A = [\text{area under } f] - [\text{area under } g]$$

or $$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]$$

Defination:- If f & g are continuous func. on $[a,b]$ & $f(x) \geq g(x)$ for all x in $[a,b]$, then the area of region bounded above by $y=f(x)$, below by $y=g(x)$, on the left by line $x=a$ & on the right of line by $x=b$ is

$$A = \int_a^b [f(x)-g(x)]\,dx$$



Example:- Find area of region bounded above by $y = x+6$, & below by $y=x^2$ on sides by $x=0$ & $x=2$.

$$\int_0^2 [f(x) - g(x)]\,dx = \int_0^2 (x+6-x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 \Rightarrow \left[\frac{2^2}{2} + 6(2) - \frac{2^3}{3}\right] - \left[\frac{0^2}{2} + 6(0) - \frac{0^3}{3}\right]$$

$$= \left(\frac{4}{2} + 12 - \frac{8}{3}\right) \Rightarrow 2+12-\frac{8}{3} \Rightarrow 14 - \frac{8}{3}$$

$$= \frac{42-8}{3} \Rightarrow \frac{34}{3} \text{ Ans.}$$

Example:- Find area of region enclosed b/w $y=x^2$ & $y=x+6$.

To find limits compare both equations

$$x^2 = x+6$$
$$x^2 - x - 6 = 0$$
$$x^2 - 3x + 2x - 6 = 0$$
$$x(x-3)+2(x-3)=0$$
$$(x-3)(x+2)=0$$

$x-3=0$ | $x+2=0$
$x=3$ | $x=-2$

So limits/interval is $[-2, 3]$

$x+6 \geq x^2$ for all values of $[-2,3]$

So $$\int_{-2}^{3} (x+6-x^2)\,dx \Rightarrow \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \left[\frac{3^2}{2} + 6(3) - \frac{(3)^3}{3}\right] - \left[\frac{(-2)^2}{2} + 6(-2) - \frac{(-2)^3}{3}\right]$$

$$= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right)$$

$$= \frac{9}{2} + 9 + 10 - \frac{8}{3} \Rightarrow \frac{9}{2} - \frac{8}{3} + 19$$

$$\Rightarrow \frac{125}{6} \text{ Ans.}$$

Example:- Find area of region enclosed by $x = y^2$ & $y = x - 2$.

$$y + 2 = x$$

comparing both

$$y^2 = y + 2$$

$$y^2 - y - 2 = 0$$

$$y^2 - 2y + y - 2 = 0$$

$$y(y-2) + 1(y-2) = 0$$

$$(y-2)(y+1) = 0$$

$y - 2 = 0$ | $y + 1 = 0$

$y = 2$ | $y = -1$

$y + 2 \geq y^2$ for all values in $[-1, 2]$.

So $$\int_{-1}^{2} (y + 2 - y^2)\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

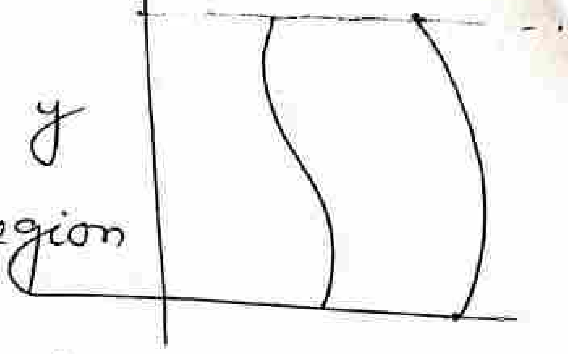
$$= \left[\frac{2^2}{2} + 2(2) - \frac{(2)^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right]$$

$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \frac{9}{2} \text{ Ans.}$$

Area b/w $x = v(y)$ & $x = w(y)$

If w & v are continuous functions & if $w(y) \geq v(y)$ for all y in $[c,d]$, then area of region bounded on left by $x = v(y)$, on right by $x = w(y)$, below by $y = c$, & above by $y = d$ is



$$A = \int_c^d [w(x) - v(x)]\, dy$$

<u>Example</u>:- Find area of region confined by $x = y^2$ & $y = x - 2$.

Solved on previous page.

lecture 34

Volume by slicing, Disks & Washers

Cyllinders

moving a 2d-circle along a line (perpendicular to circle), we get a cyllinder.

Volume of a cyllinder $= V = A.h$

A is area & h is height.

So volume of a right cyllinder is product of its area & height.

But if the cyllinder is not right, then we use method of slicing for volume of such cyllinder or object.

In this case volume will be

$$V = \lim_{\max \Delta x_k \to 0} \sum_{i=1}^{n} A(x_k^*)\, \Delta x_k = \int_a^b A(x)\,dx$$

Volume Formula:-

let S be a solid bounded by two parallel planes perpendicular to x-axis at $x=a$ & $x=b$. If, for each x in $[a,b]$ the cross-sectional area is perpendicular to x-axis is $A(x)$, then volume of Solid is $V = \int_a^b A(x)\,dx$

Similarly for y. if $y=c$ & $y=d$

$$A=\int_c^d A(y)\,dy$$

Volume of solid (Disk) perpendicular to x-axis

As $A=\pi[f(x)]^2$

$$V=\int_a^b A(x)\,dx$$

$$V=\int_a^b \pi[f(x)]^2\,dx$$

Example:- Find volume of solid that is obtained when the region under curve $y=\sqrt{x}$ over $[1,4]$ revolved about x-axis.

$$V=\int_1^4 \pi(\sqrt{x})^2\,dx$$

$$=\int_1^4 \pi x\,dx \Rightarrow \pi\int_1^4 x\,dx$$

$$=\pi\left[\frac{x^2}{2}\right]_1^4 \Rightarrow \pi\left[\frac{4^2}{2}-\frac{1^2}{2}\right]$$

$$=\pi\left(\frac{16}{2}-\frac{1}{2}\right)$$

$$=\pi\left(\frac{15}{2}\right)$$

$$V=\frac{15}{2}\pi \quad \text{Ans.}$$

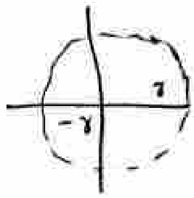
Example:- Derive the formula for the volume of a sphere of radius r.

equation of circle $x^2+y^2=r^2$

$$y^2 = r^2 - x^2$$

$$f(x) = y = \sqrt{r^2-x^2}$$

$$V = \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$



$$= \int_{-r}^{r} \pi\,(r^2-x^2)\,dx$$

$$= \pi\int_{-r}^{r} (r^2-x^2)\,dx$$

$$= \pi\left[r^2\,[x]_{-r}^{r} - \left[\frac{x^3}{3}\right]_{-r}^{r}\right]$$

$$= \pi\left[r^2(r-(-r)) - \left(\frac{r^3}{3} - \frac{(-r)^3}{3}\right)\right]$$

$$= \pi\left[2r^3 - \left(\frac{r^3}{3} + \frac{r^3}{3}\right)\right]$$

$$= \pi\left[2r^3 - \frac{2r^3}{3}\right]$$

$$= \pi\left[\frac{6r^3-2r^3}{3}\right]$$

$$V = \frac{4\pi r^3}{3}$$ Ans.

Volume of washer perpendicular to x-axis.

As
$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2$$
$$= \pi\left[(f(x))^2 - (g(x))^2\right]$$

So
$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)$$

Example:- Find volume of solid generated when region b/w graph of $f(x) = \frac{1}{2} + x^2$ & $g(x) = x$ over the interval $[0,2]$ revolved about x-axis.

$$V = \int_0^2 \pi\left(\left[\frac{1}{2} + x^2\right]^2 - [x]^2\right) dx$$
$$= \int_0^2 \pi\left[\frac{1}{4} + x^4 + x^2 - x^2\right] dx$$
$$= \int_0^2 \pi\left[\frac{1}{4} + x^4\right] dx$$
$$= \pi\left[\frac{1}{4}x + \frac{x^5}{5}\right]_0^2$$
$$= \pi\left[\frac{1}{4}(2) + \frac{(2)^5}{5}\right] - \left[\frac{1}{4}(0) + \frac{(0)^5}{5}\right]$$
$$= \pi\left[\frac{1}{2} + \frac{32}{5}\right] \Rightarrow \pi\left(\frac{5+64}{10}\right)$$
$$V = \frac{69}{10}\pi \quad \text{Ans.}$$

Volume of disk perpendicular to y-axis

$$V = \int_{c}^{d} \pi \left[u(y) \right]^2 dy$$

Volume of Washer perpendicular to y-axis:

$$V = \int_{c}^{d} \pi \left[\left(u(y) \right)^2 - \left(v(y) \right)^2 \right] dy$$

lecture 35

Volume of Cyllindrical Shells.

$$V(S) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x_k$$

$$= \int_a^b 2\pi x \, f(x) dx$$

Volume Formula:- let R be a plane region bounded above by a continuous curve $y = f(x)$ below by x-axis, & left & right by lines $x = a$ & $x = b$ respectively. Then volume of solid generated by revolving R about y-axis is given by

$$V = \int_a^b 2\pi x \, f(x) \, dx$$

Example:- Use cyllindrical shells to find the volume of solid generated by region enclosed b/w $y = \sqrt{x}$, $x = 1$, $x = 4$ & x-axis is revolved about y-axis.

$$\int_a^b 2\pi x \, f(x) dx$$

$$= \int_1^4 2\pi x \cdot \sqrt{x} \, dx \Rightarrow 2\pi \int_1^4 x^{\frac{3}{2}} dx$$

$$= 2\pi \left[\frac{x^{\frac{3}{2}+1}}{\frac{5}{2}} \right]_1^4 \Rightarrow 2\pi \times \frac{2}{5} \left(x^{5/2} \right]_1^4$$

$$= \frac{4\pi}{5} \left[4^{\frac{5}{2}} - 1^{5/2} \right] \Rightarrow \frac{4\pi}{5} \left[2^{2 \times \frac{5}{2}} - 1 \right] = \frac{124\pi}{5}$$

Surface area of cyllinder $= 2\pi x f(x)$

Example: Use Cyllindrical shells to find the volume of the solid when region R in the first quadrant enclosed b/w $y=x$ & $y=x^2$. is revolved about y-axis.

$y=x$ & $y=x^2$

comparing $x = x^2$

$x^2 - x = 0$

$x(x-1)=0$, $x=0$ or $x-1=0$

$x=1$

So limits are 0 & 1 i.e $[0,1]$ & $x > x^2$, So

$$V = \int_0^1 2\pi x \,(x - x^2)\, dx$$

$$= \int_0^1 2\pi (x^2 - x^3)\, dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= 2\pi \left(\left[\frac{1}{3} - \frac{1}{4} \right] - \left[\frac{0}{3} - \frac{0}{4} \right] \right)$$

$$= 2\pi \left(\frac{4-3}{12} \right) \Rightarrow 2\pi \left(\frac{1}{12} \right)$$

$$V = \frac{\pi}{6} \text{ Ans.}$$

Lecture 36.

Length of a plane curve

Arc Length The functions that are continuous on a given interval are called smooth functions. & their graphs are called Smooth Curves.

Generally $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$

but in functions

$$L_k = \sqrt{1 + [f'(x_k^*)]^2}\ \Delta x_k$$

$$= \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\ \Delta x_k$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

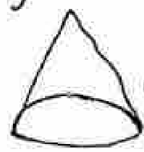
or $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Lecture 37

Area of a Surface of Revolution:-

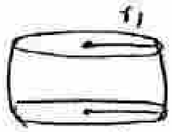
Surface generated by the polygonal path ~~around x-axis~~ is made of parts, each of which is a frustum of a cone.

A frustum of a cone is solid that you get if you chop off the pointed top of a cone



It has two radii & slanted length.

its Surface area is $S = \pi(r_1 + r_2) l$



$$r_1 = f(x_{k-1})$$

$$r_2 = f(x_k)$$

$$l = \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$

or we can also write l as arc length

$$l = \sqrt{1 + [f'(x_k^*)]^2}\ \Delta x_k$$

So

$$S = 2\pi f(x_k^*) \sqrt{1 + [f'(x_k^*)]^2}\ \Delta x_k$$

$$= \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{1 + [f'(x_k^*)]^2}\ \Delta x_k$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

Surface area formula:- Let f be a smooth non-negative function on $[a, b]$. The surface area S generated by revolving the portion of curve $y = f(x)$ b/w $x = a$ & $x = b$ about x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Example:- Find surface area of portion of sphere generated by revolving curve

$y = \sqrt{1 - x^2}$ $\quad 0 \leq x \leq \frac{1}{2}$. about x-axis.

$$f(x) = \sqrt{1 - x^2}$$

$$f'(x) = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

$$S = \int_0^{\frac{1}{2}} 2\pi \left(\sqrt{1 - x^2}\right)\sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\, dx$$

$$= 2\pi \int_0^{\frac{1}{2}} \sqrt{1 - x^2}\sqrt{1 + \frac{x^2}{1 - x^2}}\, dx$$

$$= 2\pi \int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, \frac{\sqrt{1 - x^2 + x^2}}{\sqrt{1 - x^2}}\, dx$$

$$= 2\pi \int_0^{\frac{1}{2}} 1\, dx \Rightarrow 2\pi [x]_0^{\frac{1}{2}} \Rightarrow 2\pi\left(\frac{1}{2}\right) \Rightarrow \pi \text{ Ans.}$$

Lecture 38

Work & Indefinite Integral

Workdone = Force x distance

$$W = F.d$$

unit of distance is meter (m) & units for Force are Pounds (lbs), Dynes or Newton (N).

As $F = ma$

One Dyne = Force needed to give a mass of 1 gram and acceleration of 1 cm/s^2

1 Newton = Force needed to give a mass of 1 kg and acceleration of 1 m/s^2

Common units for Workdone are

=> foot - pounds (ft lbs)

=> dyne - centimeters (dyne cm)

=> newton - meter (Nm)

One newton - meter is also called a Joule (J).

Example:- An object moves 5ft along a line while subjected to a Force of 100 lbs in its direction of motion. Find workdone.

$d =$ distance $= 5$ ft

$F =$ Force $= 100$ lbs

$$W = F.d = (100)(5) = 500 \text{ ft. lbs.}$$

This formula is used for constant force. But if force is variable then we use Calculus.

Workdone of one interval $= W_k = F(x_k^*)\,\Delta x_k$

Workdone over whole interval $= \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} F(x_k^*)\Delta x_k$

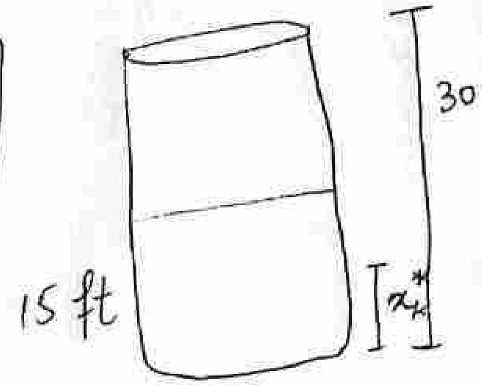
$$= \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} F(x_k^*)\,\Delta x_k$$

Defination:- If an object moves in the +ve direction over interval $[a, b]$ while subjected to a variable Force $F(x)$ in direction of motion then the workdone by force is

$$W = \int_a^b F(x)\,dx$$

Example:- A cyllindrical water tank of radius 10ft & height 30ft is half filled with water. How much force is needed to pump all the water over upper rim of tank?

$$\begin{bmatrix}\text{Force to}\\ \text{move layer}\end{bmatrix} = (\pi r^2 \Delta x_k)\begin{bmatrix}\text{weight density}\\ \text{of water}\end{bmatrix}$$



$$= (\pi(10)^2 \Delta x_k)\,(62.4)$$

$$= 6240\,\pi\,\Delta x_k$$

Now $W_k = (30 - x_k^*)(6420\,\pi\,\Delta x_k)$

$$W = \int_a^b F(x)\,dx$$

$$= \int_0^{15} (30-x)(6420\,\pi)\,dx$$

$$= 6420\,\pi \int_0^{15} (30-x)\,dx$$

$$= 6420\,\pi \left[30x - \frac{x^2}{2}\right]_0^{15}$$

$$= 6420\,\pi \left[\left(30(15) - \frac{(15)^2}{2}\right) - \left(30(0) - \frac{0^2}{2}\right)\right]$$

$$= \cancel{6420\pi}$$

$$= 2{,}106{,}000\,\pi \text{ ft/lb.}$$

$$= 6{,}616{,}194 \text{ ft/lb.}$$

Pressure:-

It is defined as Force per unit area.

$$P = \frac{F}{A} = \rho h.$$

where ρ is weight density & h is depth.

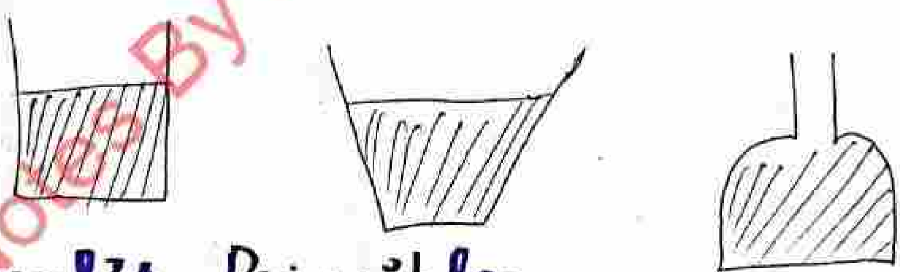
Fluid Pressure:-

If Area A is submerged horizontally in a fluid at depth h, then fluid exerts a Force F perpendicular to surface.

$$F = \rho h A.$$

ρ is weight density (ie 62.4 lbs/ft³.)

It is a physical fact that the Force F does not depend on the shape of Container.



Pascal's Principle:-

By Pascal's Principal fluid Pressure at every point of liquid is same.

Formula for Fluid Force:-

If a flat surface is immersed vertically in a liquid of weight density ρ & that submerged portion extends from $x=a$ to $x=b$. on vertical x-axis.

Total Fluid Pressure

$$F = \int_a^b \rho h(x)\, w(x)\, dx$$

Lecture 39

Improper Integral

If interval is finite $[a,b]$, then definite integral is $\int_a^b f(x)dx$. But if interval is infinite $[a,\infty)$, then integral will be

$$\int_a^{+\infty} f(x)dx = \lim_{l\to\infty} \int_a^l f(x)dx$$. called improper integral

⇒ If this limit exists then improper integral converges, & value of limit is assigned to integral.

⇒ If limit does not exist then improper integral diverges & no finite value is assigned.

⇒ The integral $\int_a^l f(x)\,dx$ represents the area under curve of $f(x)$ over $[a, l]$

Example:- Evaluate $\int_1^{+\infty} \frac{dx}{x^2}$

$$\lim_{l\to+\infty} \int_1^l \frac{1}{x^2}\,dx \Rightarrow \lim_{l\to+\infty} \int_1^l x^{-2}\,dx$$

$$= \lim_{l\to+\infty} \left[\frac{x^{-1}}{-1}\right]_1^l \Rightarrow \lim_{l\to+\infty} \left[-\frac{1}{x}\right]_1^l \Rightarrow \lim_{l\to+\infty} \left[\frac{-1}{l} - \left(\frac{-1}{1}\right)\right]$$

$$= \lim_{l\to+\infty} \left[-\frac{1}{l} + 1\right] \Rightarrow \frac{-1}{\infty} + 1 \Rightarrow 0 + 1 \Rightarrow +1 \text{ Ans}$$

Example:- Evaluate $\int_1^{+\infty} \frac{dx}{x}$

$$\lim_{l\to+\infty} \int_1^l \frac{1}{x}\,dx \Rightarrow \lim_{l\to+\infty} \left[\ln x\right]_1^l$$

$$= \lim_{l\to\infty} \left[\ln(l) - \ln(1)\right] \Rightarrow \lim_{l\to+\infty} \ln(l)$$

$$= \ln(+\infty) \Rightarrow +\infty \text{ Ans.}$$

here we get a divergent limit (bcz limit does not exist). So we are unable to find area under curve.

Find vol of a disk of radius $f(x)$ & $f(x) = \frac{1}{x}$. interval is $[1, +\infty)$

$$\int_1^{+\infty} \pi [f(x)]^2 dx = \lim_{l \to +\infty} \int_1^l \pi [f(x)]^2 dx$$

$$= \lim_{l \to +\infty} \int_1^l \pi \left(\frac{1}{x}\right)^2 dx \Rightarrow \pi \lim_{l \to \infty} \int_1^l x^{-2} dx$$

$$= \pi \lim_{l \to \infty} \left[\frac{x^{-1}}{-1}\right]_1^l \Rightarrow \pi \lim_{l \to \infty} \left[\frac{-1}{x}\right]_1^l$$

$$= \pi \lim_{l \to \infty} \left(\frac{-1}{l} - \left(\frac{-1}{1}\right)\right) \Rightarrow \pi \left(\frac{-1}{\infty} + 1\right)$$

$$= \pi (0+1) = \pi(1) = \pi \quad \underline{\text{Ans.}}$$

⇒ We can also have an improper integral of type $\int_{-\infty}^{b} f(x) dx = \lim_{l \to -\infty} \int_l^b f(x) dx$

⇒ If f is continuous on $[a, b)$ but does not have a limit from the left then we define improper integral as $\int_a^b f(x) dx = \lim_{l \to b^-} \int_a^l f(x) dx$

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⇒ If f is continuous on $[a,b]$ but fails to have a limit as x approaches a from right, we we define improper integral as

$$\int_a^b f(x)\,dx = \lim_{l \to a^+} \int_l^b f(x)\,dx$$

⇒ If both improper integrals $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ converge then $\int_a^b f(x)\,dx$ also converges as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \because (a<c<b)$$

Example:- Evaluate $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}}$

It approaches to infinity as $x \to 2$, So

$$\int_1^4 \frac{1}{(x-2)^{2/3}}\,dx = \int_1^2 \frac{1}{(x-2)^{2/3}}\,dx + \int_2^4 \frac{1}{(x-2)^{2/3}}\,dx$$

$$= \lim_{l \to 2^-} \int_1^l \frac{1}{(x-2)^{2/3}}\,dx + \lim_{l \to 2^+} \int_l^4 \frac{1}{(x-2)^{2/3}}\,dx$$

$$= \lim_{l \to 2^-} \int_1^l (x-2)^{-2/3}\,dx + \lim_{l \to 2^+} \int_l^4 (x-2)^{-\frac{2}{3}}\,dx$$

$$= \cancel{\lim_{l\to 2^-} \int_1^l}$$

$$= \lim_{l\to 2^-} \frac{\left[(x-2)^{\frac{1}{3}}\right]_1^l}{1/3} + \lim_{l\to 2^+} \frac{\left(\left[x-2\right]^{\frac{1}{3}}\right)_l^4}{1/3}$$

$$= 3\lim_{l\to 2^-}\left((l-2)^{\frac{1}{3}} - (1-2)^{\frac{1}{3}}\right) + 3\lim_{l\to 2^+}\left[(4-2)^{\frac{1}{3}} - (l-2)^{\frac{1}{3}}\right]$$

$$= 3\left[(2-2)^{\frac{1}{3}} - (-1)^{\frac{1}{3}}\right] + 3\left[(2)^{\frac{1}{3}} - (2-2)^{\frac{1}{3}}\right]$$

$$= 3(1) + 3\sqrt[3]{2}$$

$$= 3 + 3\sqrt[3]{2} \Rightarrow 3\left(1 + \sqrt[3]{2}\right)$$ Ans

Lecture 40

L'Hopital Rule & Indeterminate forms:-

$$\lim_{x\to 2} \frac{x^2-4}{x-2} \Rightarrow \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x\to 2} (x+2) \Rightarrow 2+2 \Rightarrow 4 \quad \underline{\text{Ans.}}$$

Indeterminate forms:-

$$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, 0^0, \infty^0, 1^\infty, \infty \cdot \infty$$

Theorem:- L-Hopital rule for 0/0 form

Let $\lim f(x) = 0$ & $\lim g(x) = 0$, if $\lim\left[\frac{f'(x)}{g'(x)}\right]$ has a finite value L, or if limit is $+\infty$ or $-\infty$, Then $\lim\left[\frac{f(x)}{g(x)}\right] = \lim\left[\frac{f'(x)}{g'(x)}\right]$

Example:-

Use L'hopital rule To evaluate

(i) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2}\left(\frac{2x}{1}\right)$

$$= \lim_{x\to 2} 2x \Rightarrow 2(2) \Rightarrow 4 \quad \underline{\text{Ans.}}$$

(ii) $\lim_{x\to 0} \frac{\sin 2x}{x}$

(a) L'Hopital Rule

$$\lim_{x\to 0} \frac{\frac{d}{dx}(\sin 2x)}{\frac{d}{dx}(x)} \Rightarrow \lim_{x\to 0} \frac{\cos 2x\cdot(2)}{1}$$

$$= \lim_{x\to 0} (2\cos 2x) \Rightarrow \lim 2\cos 2(0)$$

$$= \cancel{\lim}\ 2\cos 0^\circ \Rightarrow \cancel{\lim}\ 2(1) \Rightarrow 2 \quad \text{Ans.}$$

(b) algebrically:-

$$\lim_{x\to 0} \frac{\sin 2x}{x} \qquad \because \lim_{x\to 0} \frac{\sin x}{x} = 1$$

multiply & divide by 2

$$= \lim_{x\to 0} 2\left[\frac{\sin 2x}{2x}\right] \Rightarrow 2 \lim_{x\to 0}\left[\frac{\sin 2x}{2x}\right]$$

$$= 2(1) \Rightarrow 2 \quad \text{Ans.}$$

Example:- Evaluate $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$

as $1-\sin\frac{\pi}{2} \Rightarrow 1-1=0$

& $\cos\frac{\pi}{2} \Rightarrow 0$ so it is $\frac{0}{0}$ form,

we'll apply L'hopital Rule.

$$\lim_{x\to\frac{\pi}{2}} \frac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx}(\cos x)} \Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x} \Rightarrow \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}$$

$$\Rightarrow \frac{0}{1} \Rightarrow 0 \quad \text{Ans.}$$

Theorem:-

L'Hopital Rule for $\frac{\infty}{\infty}$ form:-

Suppose $\lim f(x) = \infty$ & $\lim g(x) = \infty$. If $\lim \left[\frac{f'(x)}{g'(x)}\right]$ has a finite value L, or if this limit is $+\infty$ or $-\infty$. Then

$$\lim \left[\frac{f(x)}{g(x)}\right] = \lim \left[\frac{f'(x)}{g'(x)}\right]$$

for example $\lim\limits_{x \to 0} \frac{\ln x}{\csc x} = \frac{\infty}{\infty}$

Example:- $\lim\limits_{x \to \infty} \frac{x}{e^x}$

As $\lim\limits_{x \to \infty} (x) = \infty$ & $\lim\limits_{x \to \infty} e^x = e^\infty = \infty$

So it is indeterminate form, we'll use L'hopital Rule here.

$$\lim_{x \to \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)} \Rightarrow \lim_{x \to \infty} \frac{1}{e^x} \Rightarrow \frac{1}{e^\infty}$$

$$= \frac{1}{\infty} \Rightarrow 0 \quad \underline{\text{Ans}}.$$

If $\lim f(x) = 0$ & $g(x) = \infty$ Then

$$\lim f(x) \cdot g(x) = \underline{0 \cdot \infty}$$

we can convert it into $\frac{0}{0}$ form & then solve by L' hopital rule.

$$f(x)g(x) = \frac{f(x)}{1/g(x)} = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$$

Example:- Evaluate $\lim\limits_{x \to 0} x \ln x$

$$\lim_{x \to 0} \frac{\ln x}{1/x} \Rightarrow \lim_{x \to 0} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-1})} \Rightarrow \lim_{x \to 0} \frac{\left(\frac{1}{x}\right)}{-x^{-2}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \Rightarrow \lim_{x \to 0} \left(\frac{1}{x} \times \frac{-x^2}{1}\right)$$

$$= \lim_{x \to 0} (-x) \Rightarrow 0 \quad \underline{\text{Ans.}}$$

Indeterminate forms of type 0^0, ∞^0, 1^∞ are formed by limits of the form

$$\lim_{x \to a} f(x)^{g(x)}$$

First we let them equal to a dependent variable

$$y = f(x)^{g(x)}$$

Apply $\ln$ on b/s

$$\ln y = \ln f(x)^{gx}$$

$$\ln y = g(x) \ln f(x)$$

now apply lim

$$\lim_{x\to a} \ln y = \lim_{x\to a} \left[g(x) \ln f(x) \right]$$

Example:- Show That $\lim_{x\to 0} (1+x)^{1/x} = e$

let $y = (1+x)^{\frac{1}{x}}$

$$\ln y = \ln (1+x)^{1/x}$$

$$\ln y = \frac{1}{x} \ln (1+x)$$

$$\ln y = \frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+x)}{x} \quad \because \left(\frac{0}{0} \text{ for}\right.$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} \Rightarrow \lim_{x\to 0} \frac{1}{1+x}$$

$$= \frac{1}{1+0} = 1, \quad \boxed{\ln y = 1}$$

Take anti $\ln$ on b/s $\quad e^{\ln y} = e^1 \Rightarrow e$ Ans.

Lecture # 41

Sequences & Monotone Sequences

Defination of a sequence.

⇒ Sequence in math is a succession of numbers (e-g) 2, 4, 6, 8 ..... & 1, 2, 3, 4, ...

⇒ The numbers in a sequence are called Terms of a sequence.

⇒ Each Term has a positional name (e-g) 1st term, 2nd term etc. can be written as $a_1, a_2, a_3, \ldots$

⇒ Sequences can be written as a formula

(a) $2, 4, 6, 8, \ldots$ can be $\{2n\}_{n=1}^{\infty}$

(b) $a_1, a_2, a_3, \ldots$ can b $\{a_n\}_{n=1}^{\infty}$

Formal Defination:- A sequence or infinite sequence is a function whose domain is set of positive integers, That is $\{a_n\}_{n=1}^{\infty}$ which is an alternative notation for The function $f(n) = a_n$ where $n = 1, 2, 3, \ldots$

Theorem:- A Sequence $\{a_n\}$ is said to Converge to the limit L if given any $\epsilon > 0$, There is +ve integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$\lim_{n \to +\infty} a_n = L$. A sequence that does not converge is said to be diverge.

Example:- $\lim_{n \to +\infty} \frac{n}{n+1}$

$$= \lim_{n \to +\infty} \frac{\frac{n}{n}}{\frac{n+1}{n}} \Rightarrow \lim_{n \to +\infty} \left(\frac{1}{1 + \frac{1}{n}} \right)$$

$$= \frac{1}{1 + \frac{1}{\infty}} \Rightarrow \frac{1}{1+0} \Rightarrow \frac{1}{1} \Rightarrow 1 \quad \underline{\text{Ans.}}$$

$$\lim_{n \to -\infty} \left(1 + \left(\frac{-1}{2} \right)^n \right) = 1$$

$$= \left(1 + \left(-\frac{1}{2} \right)^{-\infty} \right) \Rightarrow 1 + \frac{1}{\left(\frac{-1}{2} \right)^{\infty}} \Rightarrow 1 + \frac{1}{\infty}$$

$$= 1 + 0 = 1 \quad \underline{\text{Ans.}}$$

Theorem:- Let sequences $\{a_n\}$ & $\{b_n\}$ converge to limits L_1 & L_2 respectively. & c is a constant. Then

- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = c L_1$
- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L_1 + L_2$
- $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L_1 - L_2$
- $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = L_1 L_2$
- $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L_1}{L} \quad (L_2 \neq 0)$

Determine whether sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ converges or diverges.

$$\lim_{n\to\infty} \left(\frac{\frac{n}{n}}{\frac{2n+1}{n}}\right) = \lim_{n\to\infty} \left(\frac{1}{\frac{2n}{n} + \frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{2 + \frac{1}{n}}\right) \Rightarrow \frac{1}{2 + \frac{1}{\infty}} \Rightarrow \frac{1}{2+0} \Rightarrow \frac{1}{2}$$

Thus it converges to $\frac{1}{2}$ Ans.

Recursive Sequence:-

Some sequences are defined by specifying one or more initial terms & giving a formula that relates each subsequent term to the term that preceeds it, such sequence is called Recursive Sequence.

Example:- $a_1 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right), n \geq 1$

put $n=1$

$$a_2 = \frac{1}{2}\left(a_1 + \frac{3}{a_1}\right) \Rightarrow \frac{1}{2}\left(1 + \frac{3}{1}\right) = \boxed{2}$$

put $n=2$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{3}{a_2}\right) \Rightarrow \frac{1}{2}\left(2 + \frac{3}{2}\right) \Rightarrow \frac{1}{2}\left(\frac{4+3}{2}\right) \Rightarrow \frac{1}{2} \cdot \frac{7}{2} \Rightarrow \boxed{\frac{7}{4}}$$

put $n=3$

$$a_4 = \frac{1}{2}\left(a_3 + \frac{3}{a_3}\right) \Rightarrow \frac{1}{2}\left(\frac{7}{4} + \frac{3}{7/4}\right) \Rightarrow \frac{1}{2}\left(\frac{7}{4} + \frac{4}{7} \cdot 3\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{7}{4} + \frac{12}{7}\right) \Rightarrow \frac{1}{2}\left(\frac{49+48}{28}\right)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{97}{28} \Rightarrow \boxed{\frac{97}{56}}$$

Monotonicity & Testing for it

A sequence $\{a_n\}$ is called increasing if

$$a_1 < a_2 < a_3 < \cdots < a_n < \cdots$$

Non-decreasing if

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

decreasing if

$$a_1 > a_2 > a_3 > \cdots > a_n > \cdots$$

non-increasing if

$$a_1 \ge a_2 \ge , a_3 \ge \cdots > a_n \ge \cdots$$

$\Rightarrow$ Sequence that is either non-decreasing or non-increasing is called Monotone & which is increasing or decreasing is called Strictly ~~not~~ monotone.

$\Rightarrow$ Strictly monotone is monotone but not conversely.

Examples:-

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$ increasing

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ decreasing

strictly monotone

$1, 1, 2, 2, 3, 3, \ldots$ non-decreasing

$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$ non-increasing

all are monotone.

but $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots (-1)^{n+1}\frac{1}{n}, \ldots$

is not monotone.

Testing for monotonicity:-

(a) difference b/w successive terms:-

$a_{n+1} - a_n > 0$ Increasing

$a_{n+1} - a_n < 0$ Decreasing

$a_{n+1} - a_n \geq 0$ non-decreasing

$a_{n+1} - a_n \leq 0$ non-increasing.

(b) Ratio of successive terms:-

$\frac{a_{n+1}}{a_n} > 1$ Increasing

$\frac{a_{n+1}}{a_n} < 1$ decreasing.

By Ratio Test

$\frac{a_{n+1}}{a_n} \geq 1$ non-decreasing

$\frac{a_{n+1}}{a_n} \leq 1$ non-increasing

(c) derivative Test:-

$f'(x) > 0$ Increasing

$f'(x) < 0$ decreasing

$f'(x) \geq 0$ non-decreasing

$f'(x) \leq 0$ non-increasing

Example:- Show That $\frac{1}{2}, \frac{2}{3}, \frac{3}{4} \cdots \frac{n}{n+1} \cdots$ is an increasing sequence.

let $a_n = \frac{n}{n+1}$, $a_{n+1} = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}$

difference Test

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

So it is increasing & strictly monotone

Example:-

Show That $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$

is increasing by Ratio Test.

$a_n = \frac{n}{n+1}$ & $a_{n+1} = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}$

Now

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n+2} \div \frac{n}{n+1}$$

$$= \frac{n+1}{n+2} \times \frac{n+1}{n}$$

$$= \frac{n^2+1+2n}{n^2+2n} \Rightarrow \frac{n^2+2n+1}{n^2+2n}$$

$$= > 1$$

So it is increasing

Eventually monotonic Sequence:-

Sequence is eventually monotone if there is some integer N such that the sequence is monotone for $n \geq N$.

Example:- Show that sequence $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ is eventually decreasing.

$a_n = \frac{10^n}{n!}$, $a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

By Ratio Test

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!}$$

$$= \frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n}$$

$$= \frac{10^n \cdot 10^1}{(n+1)\,n!} \times \frac{n!}{10^n}$$

$$= \frac{10}{n+1} \quad (< 1 \text{ for all } n \geq N)$$

It is eventually decreasing for all $n \geq 10$

Theorem:- If $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$ is a non-decreasing sequence, then there are two possibilities.

(a) There's a constant M, called upper bound for the sequence, such that $a_n \leq M$ for all n in which case sequence converges to limit L satisfying $L \leq M$.

(b) No upper bound exists, in which case

$$\lim_{n \to +\infty} a_n = +\infty$$

Theorem:- If $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$ is a non-increasing sequence, then there are two possibilities:

(a) There is a constant M called a lower bound for the sequence such that $a_n \geq M$ for all n, in which case the sequence converges to limit L satisfying $L \geq M$

(b) No lower bound exist, in which case
$$\lim_{n \to +\infty} a_n = -\infty$$

Example:- Show that sequence $\left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty}$ converges.

as proved earlier it is increasing & >1 so it converges.

bcz
$$\lim_{n \to \infty} \left(\frac{\frac{n}{n}}{\frac{n+1}{n}} \right) = \lim_{n \to \infty} \left(\frac{1}{1+\frac{1}{n}} \right)$$

$= 1$ Ans.

convergence or divergence does not depend on initial value. it depends on tail end.

Lecture 42

Infinite series:-

Infinite series is an expansion that can be written in the form $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$

$u_1, u_2, u_3, \cdots$ are called terms of the series.

To add infinitely many terms is impossible So we use formula by using limits.

For $0.33333\ldots$

$$= 0.3 + 0.03 + 0.003 + \cancel{0}.0003 + \cdots$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

$$S_1 = \frac{3}{10} = 0.3$$

$$S_2 = \frac{3}{10} + \frac{3}{10^2} = 0.3 + 0.03 = 0.33$$

$$S_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 = 0.333$$

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} \quad \text{—— (1)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}$$

$$\frac{1}{10} S_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^{n+1}} \quad \text{—— (2)}$$

(1) − (2)

$$S_n - \frac{1}{10}S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} - \frac{3}{10^2} - \frac{3}{10^3} - \cdots - \frac{3}{10^n} - \frac{3}{10^{n+1}}$$

$$\left(\frac{10-1}{10}\right)S_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\frac{9}{10}S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

Now

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{10}{9}\cdot\frac{3}{10}\left(1-\frac{1}{10^n}\right)$$

$$= \frac{1}{3}\left(1-\frac{1}{\infty}\right) = \frac{1}{3}(1-0) = \frac{1}{3} \text{ Ans.}$$

Defination:- let S_n be a sequence of partial sums of $u_1 + u_2 + \cdots + u_k + \cdots$. If the sequence converges to S, it is said to be convergent & has sum $\sum_{k=1}^{\infty} u_k = S$.

If sequence diverges it is said to be divergent & has no sum.

Example:- Determine whether the series $1-1+1-1+1-1+\cdots$ converges or diverges if converges find the sum.

$S_1 = 1$

$S_2 = 1-1 = 0$

$S_3 = 1-1+1 = 1$

$S_4 = 1-1+1-1 = 0$ & so forth. The sequence of partial sum will be $1, 0, 1, 0, 1, 0, \ldots$ — So this is a divergent sequence, so it'll have no sum.

Geometric Series:- A geometric series is one of the form $a + ar + ar^2 + \ldots + ar^{k-1} + \ldots$ Each term is obtained by multiplying the previous one by a constant number, r & this number is called the ratio of the series.

Examples are $1+2+4+6+8+\ldots$

$1+1+1+1+1+\ldots$

Theorem:- A geometric series

$a + ar + ar^2 + ar^3 + \ldots + ar^{k-1} + \ldots \quad (a \neq 0)$

converges if $|r| < 1$ & diverges if $|r| \geq 1$.

If series converges, then sum is

$$S = \frac{a}{1-r} = a + ar + ar^2 + \ldots + ar^{k-1} + \ldots$$

Example:- $5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^{k-1}} + \cdots$ is a geometric Series. Find its sum if it converges.

$r = \frac{5}{4} \div 5 \Rightarrow \frac{5}{4} \times \frac{1}{5} \Rightarrow \frac{1}{4} \Rightarrow 0.25$

$|r| < 1$, so it converges.

$$\text{Sum} = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} \Rightarrow \frac{5}{\frac{4-1}{4}} \Rightarrow \frac{20}{3} \text{ Ans.}$$

Harmonic Series:- A harmonic Series is of the type $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

It seems it converges bcz every next term is smaller than previous but actually it diverges.

There are some tests, we apply on k^{th} term to find divergence/convergence.

Theorem:- Divergence Test

$\Rightarrow$ If $\lim_{k \to +\infty} u_k \neq 0$, then the series $\sum u_k$ diverges.

$\Rightarrow$ If $\lim_{k \to +\infty} u_k = 0$ then series $\sum u_k$ may either converge or diverge.

If series $\sum u_k$ converges, then $\lim_{k \to \infty} u_k = 0$

Example:- $\sum_{k=1}^{\infty} \frac{k}{k+1}$

$$= \sum_{k=1}^{\infty} \frac{\frac{k}{k}}{\frac{k}{k}+\frac{1}{k}} \Rightarrow \lim_{k \to \infty} \frac{1}{1+\frac{1}{k}}$$

$$= \frac{1}{1+\frac{1}{\infty}} \Rightarrow \frac{1}{1+0} \Rightarrow \frac{1}{1} \Rightarrow 1 \neq 0$$

So it diverges.

Theorem: (a) If $\sum u_k$ & $\sum v_k$ are convergent series & sums of these series are related by

$$\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) If c is non-zero constant, then $\sum_{k=1}^{\infty} u_k$ & $\sum_{k=1}^{\infty} cu_k$ both converge or both diverge. In case of convergence

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(c) Convergence or divergence is uneffected by deleting a finite number of terms from a series, for any +ve integer k, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_k + u_{k+1} + u_{k+2} + \cdots$$

Example:- Find sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

$$= \sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}}$$

To check convergence

$$\sum_{k=1}^{\infty}\frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

$$r = \frac{3}{4^2} \div \frac{3}{4} \Rightarrow \frac{3}{4^2} \times \frac{4}{3} \Rightarrow \frac{1}{4} = 0.25 < 1$$

So it is convergent.

$$\sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = \frac{2}{5^0} + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

$$r = \frac{2}{5} \div 2 \Rightarrow \frac{2}{5} \times \frac{1}{2} \Rightarrow \frac{1}{5} = 0.2 < 1$$

So it also converges.

$$\sum_{k=1}^{\infty} \frac{3}{4^k} - \sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = \frac{\frac{3}{4}}{1-\frac{1}{4}} - \frac{2}{1-\frac{1}{5}}$$

$$= \frac{\frac{3}{4}}{\frac{4-1}{4}} - \frac{2}{\frac{5-1}{5}}$$

$$= \frac{3}{4} \times \frac{4}{3} - 2 \times \frac{5}{4}$$

$$= 1 - \frac{5}{2} \Rightarrow \frac{2-5}{2} \Rightarrow -\frac{3}{2} \text{ Ans.}$$

Theorem:- (Integral Test)

Let $\sum u_k$ be a series with +ve terms & let $f(x)$ be the function that results when k is replaced by x in the formula of u_k.

If f is decreasing on $[a, +\infty]$, then

$$\sum_{k=1}^{\infty} u_k \quad \& \quad \int_a^{\infty} f(x)\,dx$$

both converge or both diverge.

Note:- Integral represent the continuous phenomenon & summation represent discrete one.

Example:- Use integral test to determine whether $\sum_{k=1}^{\infty} \frac{1}{k}$ converges or diverges.

Replace k by x & get $f(x) = \frac{1}{x}$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \int_{1}^{+\infty} \frac{1}{x}\,dx = \lim_{l \to +\infty} \int_{1}^{l} \frac{1}{x}\,dx$$

$$= \lim_{l \to \infty} \left[\ln x\right]_1^l = \lim_{l \to \infty} \left[\ln(l) - \ln(1)\right]$$

$$= \cancel{\lim_{l \to \infty}} \left[\ln \infty - 0\right] \Rightarrow \ln \infty \Rightarrow +\infty$$

Integral diverges, so ~~seq~~ series also diverges.

P- Series:- P-series or hyper-harmonic series is an infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

If $p = 1$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

If $p = 1/2$

$$\sum^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

Theorem:- (Convergence of p-series)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges if $p > 1$ & diverges if $0 < p \leq 1$

Example:- $1 + \frac{1}{\sqrt[3]{p}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$

Since $p = \frac{1}{3} < 1$ so The series diverges.

Lecture 43

Additional Convergence Test

Theorem:-

(The comparison Test)

Let $\sum a_k$ & $\sum b_k$ be series with nonnegative terms & suppose That

$a_1 \leq b_1,\ a_2 \leq b_2,\ a_3 \leq b_3 + \cdots + a_k < b_k + \cdots$

$\Rightarrow$ If bigger series $\sum b_k$ converges Then smaller Series $\sum a_k$ also converges.

$\Rightarrow$ If bigger series $\sum b_k$ diverges then smaller series $\sum a_k$ also diverges.

Theorem:- The Ratio Test:-

let $\sum u_k$ be a series with +ve terms & suppose that $\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$

(a) If $\rho < 1$, the series converges.

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

(c) If $\rho = 1$, the series may converge or diverge so that another test must be tried.

Example:- **Use Ratio test to determine convergence or divergence.**

a) $\sum_{k=1}^{\infty} \frac{1}{k!}$ here $u_k = \frac{1}{k!}$ & $u_{k+1} = \frac{1}{(k+1)!}$

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{1/(k+1)!}{1/k!} \Rightarrow \lim_{k \to +\infty} \frac{1}{(k+1)!} \times \frac{k!}{1}$$

$$= \lim_{k \to \infty} \frac{k!}{(k+1)!} \Rightarrow \lim_{k \to +\infty} \frac{k!}{(k+1)k!} \Rightarrow \lim_{k \to +\infty} \frac{1}{k+1}$$

$$= \frac{1}{\infty + 1} \Rightarrow \frac{1}{\infty} \Rightarrow 0 < 1$$ convergent series.

(b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$ here $u_k = \frac{k}{2^k}$ & $u_{k+1} = \frac{k+1}{2^{k+1}}$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1/2^{k+1}}{k/2^k} \Rightarrow \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \times \frac{2^k}{k}$$

$$= \lim_{k\to+\infty} \frac{k+1}{2^k \cdot 2} \times \frac{2^k}{k} \Rightarrow \lim_{k\to+\infty} \frac{k+1}{2k}$$

$$\Rightarrow \lim_{k\to+\infty} \left(\frac{k}{2k} + \frac{1}{2k}\right) \Rightarrow \lim_{k\to+\infty} \left(\frac{1}{2} + \frac{1}{2k}\right) \Rightarrow \frac{1}{2} + \frac{1}{\infty}$$

$$= \frac{1}{2} + 0 \Rightarrow \frac{1}{2} \Rightarrow 0.5 < 1$$

So series converges.

(c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ here $u_k = \frac{k^k}{k!}$, $u_{k+1} = \frac{(k+1)^{k+1}}{(k+1)!}$

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \times \frac{k!}{k^k}$$

$$= \lim_{k\to+\infty} \frac{(k+1)^k (k+1)}{(k+1)\, k!} \times \frac{k!}{k^k} \Rightarrow \lim_{k\to+\infty} \frac{(k+1)^k}{k^k}$$

$$= \lim_{k\to+\infty} \left(\frac{k+1}{k}\right)^k \Rightarrow \lim_{k\to+\infty} \left(\frac{k}{k} + \frac{1}{k}\right)^k$$

$$\Rightarrow \lim_{k\to+\infty} \left(1 + \frac{1}{k}\right)^k \Rightarrow e = 2.71 > 1$$

diverges

(d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$ here $u_k = \frac{(2k)!}{4^k}$ & $u_{k+1} = \frac{[2(k+1)]!}{4^{k+1}}$

$$P = \lim_{k\to+\infty} \frac{u_{k+1}}{u^k} = \frac{[2(k+1)]!}{4^{k+1}} \times \frac{4^k}{(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)!\ 4^k}{4^k \cdot 4\ \cdot (2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)(2k)!}{4\cdot(2k)!}$$

$$= \frac{1}{4} \lim_{k\to\infty} (2k+2)(2k+1)$$

$$= \frac{1}{4}\left[2(\infty)+2\right] * \left[2(\infty)+1\right] = \infty$$

Series diverges.

Example:- Determine whether the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2k-1} + \dots$$

converges or diverges.

here $u_k = \frac{1}{2k-1}$ & $u_{k+1} = \frac{1}{2(k+1)-1} = \frac{1}{2k+2-1}$

$$= \frac{1}{2k+1}$$

$$P = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \frac{1}{2k+1} \times \frac{2k-1}{1} \Rightarrow \frac{2k-1}{2k+1}$$

$$= \lim_{k\to\infty} \frac{\frac{2k-1}{k}}{\frac{2k+1}{k}} \Rightarrow \lim_{k\to\infty} \frac{\frac{2k}{k} - \frac{1}{k}}{\frac{2k}{k} + \frac{1}{k}} \Rightarrow \lim_{k\to\infty} \frac{2 - \frac{1}{k}}{2 + \frac{1}{k}}$$

$$= \frac{2 - \frac{1}{\infty}}{2 + \frac{1}{\infty}} \Rightarrow \frac{2-0}{2+0} \Rightarrow \frac{2}{2} = 1$$

So ratio test is of no use. bcz $p = 1$ series may converge or diverge. We'll apply Integral Test now.

$$\int_1^{+\infty} \frac{1}{2x-1}\, dx = \frac{1}{2} \int_1^{+\infty} \frac{1}{2x-1}\, 2dx$$

$$= \frac{1}{2} \lim_{l\to\infty} \int_1^{l} \frac{1}{2x-1}\, 2dx \Rightarrow \frac{1}{2} \lim_{l\to\infty} \Big[\ln(2x-1)\Big]_1^{l}$$

$$= \frac{1}{2} \lim_{l\to\infty} \Big[\ln(2l-1) - \ln[2(1)-1]\Big]$$

$$= \frac{1}{2} \lim_{l\to\infty} \Big[\ln(2l-1) - 0\Big]$$

$$= \frac{1}{2} \ln(2(\infty)-1) \Rightarrow \frac{1}{2}(\infty) = \infty$$

So series diverges.

Theorem:- The Root Test

Let $\sum u_k$ be series with +ve terms & suppose That $p = \lim_{k\to\infty} (u_k)^{1/k}$

(a) If $p < 1$, series converges.

(b) If $p > 1$ or $p = +\infty$, series diverges.

(c) If $p = 1$, series may converge or diverge.

Example:- $\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$

since $p = \lim_{k\to\infty} (u_k)^{1/k} = \lim_{k\to\infty} \left[\left(\frac{4k-5}{2k+1}\right)^k\right]^{\frac{1}{k}}$

$$= \lim_{k\to\infty} \left(\frac{4k-5}{2k+1}\right) \Rightarrow \lim_{k\to\infty} \left(\frac{\frac{4k}{k} - \frac{5}{k}}{\frac{2k}{k} + \frac{1}{k}}\right)$$

$$= \lim_{k\to\infty} \left(\frac{4 - \frac{5}{k}}{2 + \frac{1}{k}}\right) = \left(\frac{4 - \frac{5}{\infty}}{2 + \frac{1}{\infty}}\right) \Rightarrow \frac{4-0}{2+0}$$

$$\Rightarrow \frac{4}{2} \Rightarrow 2 > 1.$$

So series diverges.

Informal Principle:- Constant terms in The denominator of u_k can usually be deleted without affecting The convergence or divergence.

Example:- $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$ converges or diverges. using informal principle.

$$\sum_{k=1}^{\infty} \frac{1}{2^k+1}$$ deleting constant 1

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$ behaves alike.

If first is convergent then second also converges.

Example:- Use above principle to help guess whether the following series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+2k}}$ converge or diverge.

Deleting $2k$ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}}$

$$= \sum_{k=1}^{\infty} \frac{1}{(k^3)^{\frac{1}{2}}} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

As $p = \frac{3}{2} > 1$ So it converges.

The origional Series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+2k}}$ also converges.

Theorem:- (Limit Comparison Test.)

Let $\sum a_k$ & $\sum b_k$ be series with +ve terms & suppose that $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$.

If ρ is finite & $\rho > 0$, then series both converges or both diverges.

Example:- Use limit comparison to test whether $\sum_{k=1}^{\infty}\left(\frac{1}{2k^2-k}\right)$ converges or diverges

deleting k

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} \Rightarrow \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$a_k = \frac{1}{2k^2-k} \quad \& \quad b_k = \frac{1}{2k^2}$$

As $\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{2k^2-k} \div \frac{1}{2k^2}$

$$= \lim_{k\to\infty} \frac{1}{2k^2-k} \times \frac{2k^2}{1}$$

$$= \lim_{k\to\infty} \left(\frac{2k^2}{2k^2-k}\right)$$

$$= \lim_{k\to\infty} \left(\frac{2k^2/k^2}{\frac{2k^2-k}{k^2}}\right) \Rightarrow \lim_{k\to\infty} \left(\frac{2}{\frac{2k^2}{k^2}-\frac{k}{k^2}}\right)$$

$$= \lim_{k\to\infty} \left(\frac{2}{2-\frac{1}{k}}\right) \Rightarrow \frac{2}{2-\frac{1}{\infty}} \Rightarrow \frac{2}{2-0}$$

$$= \frac{2}{2} \Rightarrow 1$$

As p is finite & +ve So series converge.

Comparison Test:-

To apply This test we need two steps.

$\Rightarrow$ Guess whether sequence converges or diverge.

$\Rightarrow$ Find a series That prove the guess true.

If guess is divergent find a divergent series whose terms are smaller that Σu_k & if guess is convergent find a convergent series whose terms are bigger Than Σu_k.

<u>Examples:-</u> $\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{4}}$

Use comparison test to find convergence or divergence.

As $\frac{1}{k-\frac{1}{4}} > \frac{1}{k}$

It behaves like divergent harmonic series

So we'll find a divergent series smaller than $\frac{1}{k-\frac{1}{4}}$. We can do this by deleting $-\frac{1}{4}$. So we get $\sum_{k=1}^{\infty} \frac{1}{k}$.

comparison test

$$P = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \left(\frac{1}{k-\frac{1}{4}} \times \frac{k}{1}\right)$$

$$= \lim_{k\to+\infty} \left(\frac{k}{\frac{4k-1}{4}}\right) = \lim_{k\to+\infty} \left(\frac{4k}{4k-1}\right)$$

$$= \lim_{k\to+\infty} \left(\frac{\frac{4k}{k}}{\frac{4k-1}{k}}\right) \Rightarrow \lim_{k\to+\infty} \left(\frac{4}{4-\frac{1}{k}}\right)$$

$$= \frac{4}{4-\frac{1}{\infty}} \Rightarrow \frac{4}{4-0} \Rightarrow \frac{4}{4} \Rightarrow 1 \text{ (finite \& +ve)}$$

So it converges so origional series also converges.

Lecture 44

Alternating Series, Conditional convergence

- $1 - 1 + 1 - 1 + \cdots$
- $1 + 2 - 3 + 4 - 5 + 6 + \cdots$

Most generally, an alternating series has one of the two following forms.

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots \quad \text{——} \; ①$$

$$\sum_{k=1}^{\infty} (-1)^{k} a_k = -a_1 + a_2 \cancel{+} - a_3 + a_4 - \cdots \quad \text{——} \; ②$$

Theorem:- (Alternating Series Test)

An alternating series of either form ① or ② converges if following two conditions are satisfied.

(a) $a_1 > a_2 > a_3 \cdots\cdots > a_k > \cdots$

(b) $\lim\limits_{k \to \infty} a_k = 0$

Example:- Use the alternating series test to show that the following series converge

$$\sum_{K=1}^{\infty} (-1)^{K+1} \frac{1}{K}$$

$$a_k = \frac{1}{k} \quad , \quad a_{k+1} = \frac{1}{k+1}$$

here $\underline{a_k > a_{k+1}}$

& $\lim_{k \to +\infty} a_k = \lim_{k \to +\infty} \frac{1}{k} \Rightarrow \frac{1}{\infty} \Rightarrow \underline{0}$

Both conditions are satisfied, so this series is convergent. Note that it is harmonic series, but alternating. It is called "Alternating Harmonic Series."

<u>Harmonic Series diverges, alternating harmonic Series converges.</u>

Theorem:- If an alternating series satisfies the conditions of alternating ~~sequence~~ series test, & if the sum S of the series is approximated by n^{th} partial sum S_n, thereby resulting in an error $S-S_n$. Then

$|S-S_n| < a_{n+1}$. Sign of error is same as that of the co-efficient of a_{n+1} in the series.

Example:- The alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{k+1}\frac{1}{k}+\cdots$$

$$S_7 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7} = \frac{319}{420}$$

&

$$S_8 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8} = \frac{533}{840}$$

So

$$\frac{533}{840} < S < \frac{319}{420}$$

If we take $S = \ln 2$.

$$\frac{533}{840} < \ln 2 < \frac{319}{420}$$

$$0.6345 < \ln 2 < 0.7576$$

$$0.6345 < 0.6931 < 0.7576$$

$$|\ln 2 - S_7| = \left|\ln 2 - \frac{319}{420}\right|$$

$$= |0.6931 - 0.7576|$$

$$= 0.0663 < a_8 = \frac{1}{8}$$

$$|\ln 2 - S_8| = \left|\ln 2 - \frac{533}{840}\right|$$

$$= |0.6931 - 0.6345|$$

$$= 0.0586 < a_9 = \frac{1}{9}$$

Absolute & Conditional Convergence:-

The series $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$ has mixed signs & is not any of categories studied above. Now we'll see some convergence result to apply on such type of series.

Defination:- A series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

is said to converge absolutely, if series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \cdots + |u_k| + \cdots$$

converges.

Example:- $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$

As series of absolute value

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$$

is convergent geometric Series,

So the origional series also converges.

Example:- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$ is an alternating harmonic series does not converge absolutely bcz $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges.

If a series converges absolutly, Then it converges.

Example:- Show That The series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges.

Since $|\cos k| \leq 1$ for all k

$$\left|\frac{\cos k}{k^2}\right| \leq \frac{1}{k^2}$$

Thus $\left|\sum_{k=1}^{\infty} \frac{\cos k}{k^2}\right|$ converges by comparison test, AND consequently $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges.

If $\sum |u_k|$ diverges, no conclusion can be drawn about convergence or divergence of $\sum u_k$

for example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots$

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \cdots \frac{-1}{k} - \cdots$$

both are harmonic series, first is alternating & converging. but 2nd is diverging.

Yet absolute harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} + \cdots$$ is diverging.

A series which is convergent but not absolutely convergent, is called Conditionally convergent.

Theorem:- (Ratio Test for absolute convergence)

Let $\sum u_k$ be a series with non-zero terms & suppose that $p = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|}$

(a) If $p<1$, the series $\sum u_k$ converges absolutely & therefore converges.

(b) If $p>1$ or $p=+\infty$, then series $\sum u_k$ diverges.

(c) If $p=1$, no conclusion can be drawn.

Example:- $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$ converges or diverges?

$$u_k = \frac{2^k}{k!} \quad \& \quad u_{k+1} = \frac{2^{k+1}}{(k+1)!}$$

$$p = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \times \frac{k!}{2^k}$$

$$= \lim_{k\to\infty} \frac{2^k \cdot 2 \cdot k!}{(k+1)(k)!\, 2^k} \Rightarrow \lim_{k\to\infty} \frac{2}{k+1} \Rightarrow \frac{2}{\infty} \Rightarrow 0 < 1$$

converges

Power Series in x

If $c_0, c_1, c_2 \ldots\ldots$ are constants & x is a variable

Then a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots$$

is called a power series in x.

Examples:-

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cancel{+} - \frac{x^6}{6!} + \cdots$$

Theorem:- For any power series in x, exactly one of the following is true.

The series converges only for $x = 0$

The series converges absolutely (& hence converges) for all real values of x.

The series converges absolutely (& hence converges) for all x in finite open interval $(-R, R)$ & diverges $-R<x<R$

Radius & Interval of convergence:-

Set of values for which power series converges is always an interval centered at 0. we call this the interval of convergence, corresponding to this interval series has radius called radius of convergence.

Diverge $\quad$ 0 $\quad$ Diverge $\quad$ (Radius of convergence $R = 0$)

convergence $\quad$ 0 $\quad$ (Radius of convergence $R = +\infty$)

divergence $\quad$ convergence $\quad$ 0 $\quad$ divergence $\quad$ (Radius of convergence R)

Example:- Find interval of convergence & radius of convergence of $\sum_{k=1}^{\infty} x^k$.

Ratio test for absolute convergence

$$\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| \Rightarrow \lim_{k\to\infty}\left|\frac{x^k x}{x^k}\right|$$

$$= \lim_{k\to\infty}|x| = |x|$$

So it conveges absolutely if $\rho = |x| < 1$ & diverges if $\rho = |x| > 1$. no result if $|x| = 1$

(i-e) $|x| = 1$ means $x = \pm 1$

$x = 1$ or $x = -1$

convergence will be investigated separately at both points.

$$\sum_{k=0}^{\infty} (1)^k = 1+1+1+1+1+\cdots$$

$$\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+1-1+\cdots$$

both diverge. The interval is $(-1, 1)$ & $R = 1$ (radius of convergence).

Power series in $x-a$

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \cdots$$

called power series in $x-a$.

Examples
$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \cdots$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \cdots$$

Lecture 45

Taylor & Maclaurin Series

Theorem:-

Maclaurin Polynomials:-

If f can be differentiated n times at 0, Then we define The n^{th} maclaurin Polynomial for f to be

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$$\cdots + \frac{f^n(0)x^n}{n!}$$

Example:- Find maclaurin polynomials P_0, P_1, P_2, P_3 & P_n for e^x.

Let $f(x) = e^x$, $\quad f(0) = e^0 = 1$

$f'(x) = e^x$, $\quad f'(0) = e^0 = 1$

$f''(x) = e^x$, $\quad f''(0) = e^0 = 1$

$f'''(x) = e^x$, $\quad f'''(0) = e^0 = 1$

So

$P_0(x) = f(0) = 1$

$P_1(x) = f(0) + f'(0)x = 1 + 1(x) \Rightarrow 1 + x$

$P_2(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} = 1 + x + \frac{x^2}{2!}$

$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^n(0)x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

Taylor Polynomial:-

If f can be determined n times at 0, Then we define The n^{th} Taylor polynomial for f about $x=a$ to be

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^n(a)(x-a)^n}{n!}$$

In special case when $x=0$, Taylor's series for f is called Maclaurin Series for f.

Example:- Find maclaurin polynomial & maclaurin Series.

(a) e^x

Maclaurin polynomial for e^x is

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

& maclaurin series for e^x is

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$

(b) $\sin x$

$f(x) = \sin x \quad , \quad f(0) = \sin 0^\circ = 0$

$f'(x) = \cos x \quad , \quad f'(0) = \cos 0^\circ = 1$

$f''(x) = -\sin x \quad , \quad f''(0) = -\sin 0^\circ = 0$

$f'''(x) = -\cos x \quad , \quad f'''(0) = -\cos 0^\circ = -1$

$f''''(x) = -(-\sin x) \quad , \quad f''''(0) = \sin 0^\circ = 0$
$= \sin x$

So

$$P_n(x) = 0 + 1(x) + 0 + \frac{(-1)x^3}{3!} + 0 + \frac{(1)x^5}{5!} + 0 + \frac{(-1)x^7}{7!} + 0 + \frac{(1)(x^9)}{9!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

its k^{th} Term'll be

$$(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$